Examples of o-minimality in Algebraic Geometry

Hunter Spink

September 2

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The perfect setting for proving **qualitative finiteness statements**

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- $\exists x \in \mathbb{R}$, $\forall x \in \mathbb{R}$.
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- Largest known o-minimal $\mathcal{F}$ is "Pfaffian Closure" of "Restricted Analytic Functions".

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Corollary (Most important for applications)

If S_ϕ is a finite set of points, then $|S_\phi|$ is bounded above by a function of comp(ϕ).

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(Only uses S is locally parametrized with finite k'th-order derivatives).

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Key Idea: $\mathcal{F} = \{+, \times, \exp_{\mathbb{R}}, \cos(2\pi x)|_{[0,1]}, \sin(2\pi x)|_{[0,1]}\}$ is o-minimal.

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If $V \subset S \subset \mathbb{C}^2$, then S contains the ≥ 2-dimensional complexification of V by analytic continuation. Hence $S = V$ is algebraic, and one can show this can't happen when f is irreducible with $|Supp(f)| \geq 3$. □

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This can't happen with $R = 124$ being larger and larger, or it violates the height bound of e.g. $C \cdot R^{1/3}$.
Hence there are only finitely many solutions in roots of unity on $f = 0$.