Hyperkähler varieties as Brill-Noether loci on curves.

$$\mathcal{F}_g = \{ (\underbrace{X, H}_{\text{polarised K3 surface}}, C) \} \xrightarrow{c_g} \mathcal{M}_g$$

$C \in |H|$ smooth curve of gen g.

$\dim \mathcal{F}_g = 19 + g$

$\geq \dim \mathcal{M}_g = 3g - 3$

i.e. $g \leq 11$

[GLM] If $g \leq 11$, $g \neq 10$, c_g is dominant.

[Mukai, CLM] If $g \geq 11$, $g \neq 12$, c_g is birational onto its image.

[Mukai's program] to find a rational inverse of c_g. and construct the K3 surface (X, H) which contains $\rightsquigarrow$ general curve $\in \operatorname{im}(c_g)$ C.

[Mukai $\rightsquigarrow$ $g = 11$, ABS, $g \equiv 1 \pmod 4$, F $\rightsquigarrow$ for any $g \geq 11$, $g \neq 12$

Q. Can we characterize curves in $\mathrm{im}(\ell_g)$?

· Fix $(r,k) \in \mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}$ s.t $\begin{cases} k < r \\ \gcd(r,k) = 1 \end{cases}$, take $\underline{g >> 0}$ (*)

$\exists !\, s \in \mathbb{Z}$ s.t $-2 \leq k^2(2g-2) - 2rs < -2+2r$ +

take a curve C of genus $\underline{g}$

$$RN_C := BN_C(r, k(2g-2), r+s) = \left\{ \begin{array}{l} \text{rank } r \text{ semistable vector bundle on } C \\ \deg(E) = k(2g-2), \; h^0(E) \geq r+s \end{array} \right\}.$$

$\hookrightarrow$ expected dimension is negative.

any curve.

Thm [F]. Let (X,H) be a polarized K3 surface, $\mathrm{Pic}(X) = \mathbb{Z}\cdot H$, $C \in |H|$ of genus $\underline{g}$

$$\psi : M_X(v = (r, kH, s)) \longrightarrow BN_C$$

$r = \mathrm{ch}_0$, $kH = \mathrm{ch}_1$, $s = \mathrm{ch}_2 + \mathrm{ch}_0$

$$E \longrightarrow E|_C$$

smooth proj dim.

$2g - 2r\lfloor g/r \rfloor$

ψ is an isomorphism.

$g = rs + 1$

$= 2$

Cor. Any $F \in BN_C$ is stable, $\Lambda^r F = \omega_C^{\otimes k}$ and $h^0(F) = r + s$.

Q. $\forall$ $C \in \mathrm{im}(\mathrm{alg})$, $g >> 0$, $h := \max \left\{ h^0(C, F) \;\middle|\; \begin{array}{l} F: \text{rank } r\text{-stable bundle} \\ \Lambda^r F = \omega_C \end{array} \right\}$

$\rightsquigarrow$ Is $BN_C(r, \omega_C, h)$ a hyperkähler variety? (*)

Bridgeland stability condition.

(X, H), $\omega_X = \mathcal{O}_X$, $H^1(X, \mathcal{O}_X) = 0$ $\quad D(X)$

Mukai vector $v(E) = (ch_0(E), ch_1(E), ch_0(E) + ch_2(E)) \in N(X)$
$$\begin{gathered} \| \\ \mathbb{Z} \oplus NS(X) \oplus \mathbb{Z}. \end{gathered}$$

$\underline{Coh(X)} \rightsquigarrow \mu_H(E) = \begin{cases} \dfrac{H ch_1}{H^2 ch_0} & ch_0 \neq 0 \\ +\infty. & \text{oth.} \end{cases}$

new abelian category. new slope.

$E \in Coh(X)$ is μ_H-(semi)stable.

$0 \neq F \subset E$ $\quad \mu(F) (\leq) \mu(E/F)$

HN filtration $\rightsquigarrow \mu_H^{\mp}(E)$: max/min filtration.

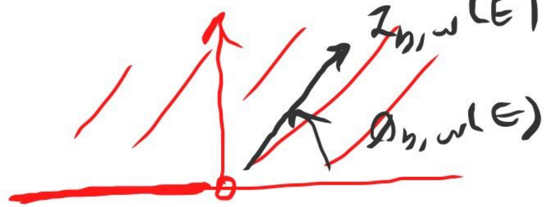
Fix $b \in \mathbb{R}$, $\mathrm{Coh}^b(X) = \{ E^{-1} \xrightarrow{d} E^0 \mid \mu_H^+(\ker d) \le b \,,\ \mu_H^-(\mathrm{coker}\, d) > b \}$.

$\hookrightarrow$ [Bridgeland]. abelian subcategory.

for any ($\omega > \frac{H^2 b^2}{2}$) ; $Z_{b,\omega} : N(X) \longrightarrow \mathbb{C}$

$[E] \longrightarrow -ch_2(E) + \omega\, ch_0(E) + i\left(\frac{H ch_1(E)}{H^2} - b\, ch_0(E) \right)$

≥ 0 if $E \in \mathrm{Coh}^b(X)$



$0 < \phi_{b,\omega}(E) \le 1$

Def. $E \in \mathrm{Coh}^b(X)$ is (semi) stable if $0 \ne F \subset E$ in $\mathrm{Coh}^b(X)$

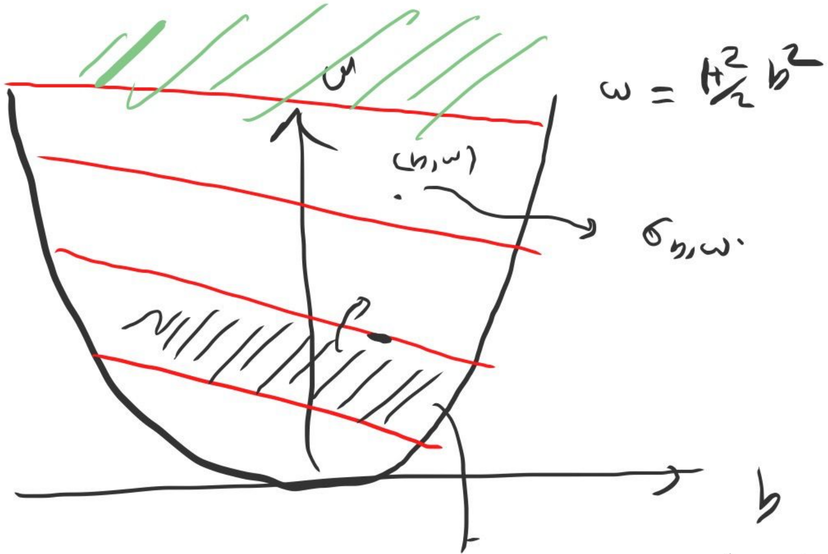
$$\phi_{b,\omega}(F) \ (\le)\ \phi_{b,\omega}(E)$$

[Bridgeland]. for any $\omega > \frac{H^2 b^2}{2}$, $\sigma_{b,\omega} = (\mathrm{Coh}^b, Z_{b,\omega})$ is a stability condition.

i.e. HN : $E \in \mathrm{Coh}^b(X)$ $\quad 0 = E_0 \subset E_1 \subset \dots \subset E_n = E$

E_i/E_{i-1} σ-semistable

$\phi(E_1/E_0) > \phi(E_2/E_1) > \dots$



$\sigma_{b,\omega}$-stability of E is unchanged.

for any $E \in D(X)$ there exists a wall and chamber decomposition.

Recall: $i : C \hookrightarrow X$

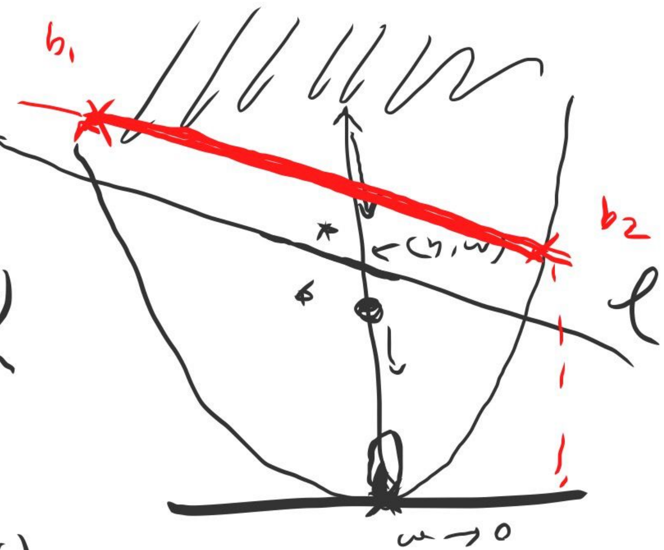
$$\Psi : M_X(v) \longrightarrow BN_C$$
$$E \longrightarrow E|_C$$

Pf. $F \in BN_C \rightsquigarrow i_* F$ is Gieseker-semistable.

Large volume limit: A coherent sheaf E is $\sigma_{b,\omega}$-semistable for $\omega \gg 0$ iff E is H-Gieseker semistable.

$\rightsquigarrow i_* F$ is $\sigma_{b,\omega}$-semistable when $\omega \gg 0$

prop. the highest wall for $i_* F$ for $\boxed{F \in BN_C}$ is the wall ℓ that sheaves with Mukai vector v are walking, and $i_* F$ gets destabilised along this wall **iff** $F = E|_C$ for $E \in M_X(v)$ and the destabilising sequence



(*) $\quad E \longrightarrow i_* E|_C \longrightarrow E(-H)[1]$ in $Coh^b(X)$

$b_2 - b_1$

$\phi_{b,w}(E) = \phi_{b,w}(E(-H)[1]) \Rightarrow i_* E|_C$ is $\sigma_{b,w}$-semistable.

$\sigma_{b,w}$-stable

$\phi_{b,w^+}(E) < \phi_{b,w^+}(E(-H)[1]) \Rightarrow i_* E|_C$ is σ_{b,w^+}-stable.

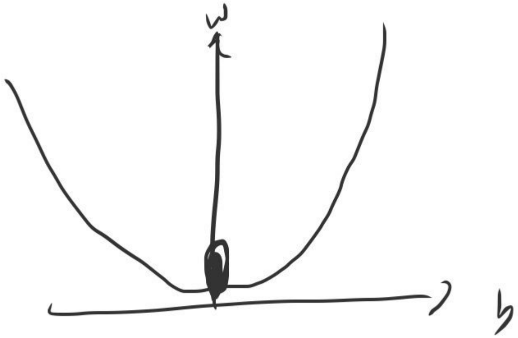
$\Rightarrow E|_C$ is slope-stable.

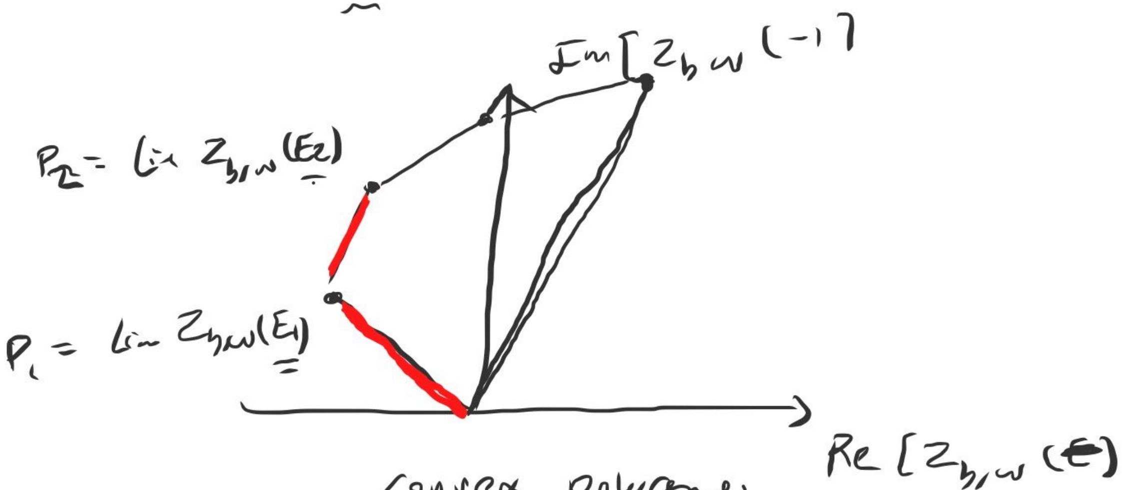
$\phi_{b,w^-}(E) > \phi_{b,w^-}(E(-H)[1]) \Rightarrow i_* E|_C$ is σ_{b,w^-}-unstable.

σ_{b,w^-}-HN filtration of $i_* E|_C$ is (*)

$\rightsquigarrow$ uniqueness of HN filtration gives Ψ is injective!

HN polygon: $\left\{ \begin{array}{l} E \in \mathrm{Coh}^b(X) \\ \sigma_{b,w} \end{array} \right.$ $0 = E_0 \subset E_1 \subset E_2 \ldots \subset E_n = E$





Convex polygon.

Prop. (BN wall) take $E \in \mathrm{Coh}^{b=0}(X)$.

If $0 < w \ll 1$, the HN filtration of E is a fixed sequence

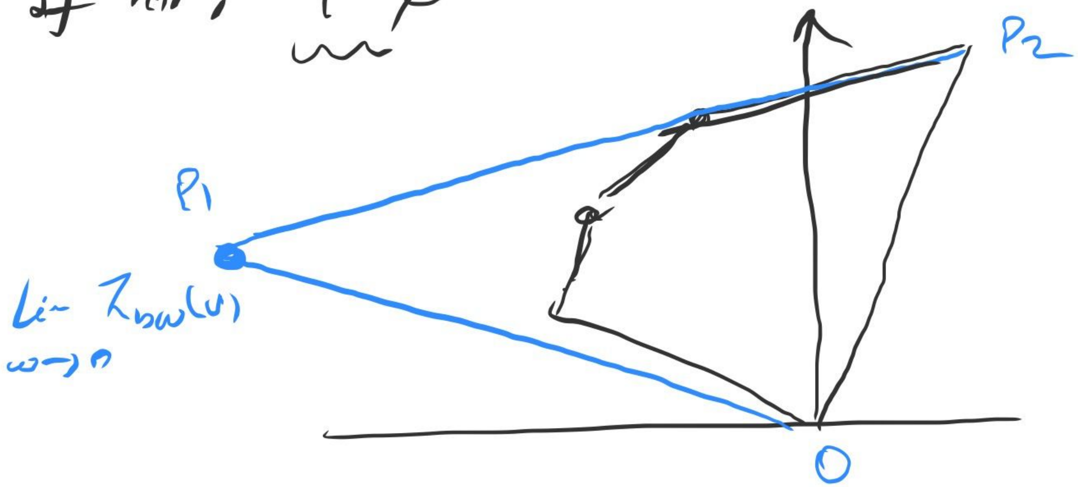
$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

$P_i := \lim_{w \to 0} Z_{b,w}(E_i)$; there exists $\|\cdot\|$ on $\mathbb{R}^2$ s.t.

$$h^0(X, E) \leq \frac{\chi(E)}{2} + \frac{1}{2} \sum_{i=1}^{n} \| P_i P_{i-1} \|$$

claim. φ is surjective.

If not, $\underset{\sim}{F} \notin \mathrm{im}(\varphi)$. $\rightsquigarrow$ HN polygon of F



$$\underline{r+s} \le h^0(i_* F) \le \frac{\chi(i_* F)}{2} + \frac{1}{2} \sum_{i=1}^{n} \| P_i P_{i-1} \|$$

$\underset{\text{prop.}}{\Downarrow}$

$$< \frac{\chi(i_* F)}{2} + \frac{1}{2} \left(\| OP_1 \| + \| P_1 P_2 \| \right)$$

$$= r+s$$

$\lightning$. $\square$