Regular centralizers and the wonderful compactification

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The wonderful compactification

G a semisimple algebraic group of adjoint type over $\mathbb{C}$

$\tilde{G}$ simply-connected cover

V regular irreducible $\tilde{G}$-representation

Definition [deConcini–Procesi]

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & (\text{End } V)\backslash\{0\} \\ \downarrow & & \downarrow \\ G & & \mathbb{P}(\text{End } V). \end{array}$$

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Definition [DeConcini–Procesi]

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & (\mathrm{End}\ V)\backslash\{0\} \\ \downarrow & & \downarrow \\ G & \overset{\rho}{\dashrightarrow} & \mathbb{P}(\mathrm{End}\ V). \end{array}$$

The wonderful compactification of G is $\overline{G} := \overline{\rho(G)}$.

- independent of V
- smooth projective $G \times G$-variety

The wonderful compactification

The boundary of $\overline{G}$ is a simple normal crossing divisor

$$D = D_1 \cup \ldots \cup D_l, \qquad l = \operatorname{rk} G.$$

$$G \times G\text{-orbits on } D \quad \longleftrightarrow \quad \text{subsets } J \subset \{1, \ldots, l\}:$$

$$\overline{\mathcal{O}_J} = \bigcap_{j \in J} D_j.$$

For each $J \subset \{1, \ldots, l\}$: parabolic subgroups P_J and P_J^-

common Levi $L_J := P_J \cap P_J^-$

$$\begin{array}{ccc} L_J/Z_{L_J} & \hookrightarrow & \mathcal{O}_J \\ & & \downarrow \\ & & G/P_J \times G/P_J^- \end{array}$$

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$$\begin{array}{ccc} \overline{L_J/Z(L_J)} & \hookrightarrow & \overline{\mathcal{O}_J} \\ & & \downarrow \\ & & G/P_J \times G/P_J^- \end{array}$$

Example

Let $G = PGL_2 \quad \rightsquigarrow \quad \tilde{G} = SL_2, \quad V = \mathbb{C}^2$. Then

$$\rho(G) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc \neq 0 \right\},$$

and $\overline{G} = \mathbb{P}(M_{2\times 2}) \cong \mathbb{P}^3$.

$$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Non-example

Let $G = PGL_n$ for $n \geqslant 3$. Then $V = \mathbb{C}^n$ is not a regular rep of $\tilde{G} = SL_n$, and

$$\overline{G} \ncong \mathbb{P}^{n^2-1}.$$

Regular centralizers

The regular locus of $\mathfrak{g} = \operatorname{Lie} G$ is

$$\mathfrak{g}^{\mathrm{r}} = \{x \in \mathfrak{g} \mid \dim G^x = l\}.$$

Example: $\mathfrak{g} = \mathfrak{sl}_n$

regular $\Leftrightarrow$ Jordan blocks have distinct eigenvalues

- x regular semisimple $\rightsquigarrow$ G^x is a maximal torus
- x regular nilpotent $\rightsquigarrow$ G^x is a abelian group $\cong \mathbb{C}^l$

Regular centralizers

Fix a maximal torus $T \subset G$

$$\rightsquigarrow \overline{T} \subset \overline{G} \quad \text{is the Coxeter toric variety.}$$

Let $\{e, h, f\} \subset \mathfrak{g}^{\mathrm{r}}$ be an $\mathfrak{sl}_2$-triple.

Theorem [Kostant]

The Kostant slice

$$\mathcal{S} := f + \mathfrak{g}^{e} \subset \mathfrak{g}^{\mathrm{r}}$$

meets each regular G-orbit on $\mathfrak{g}$ exactly once and transversally.

Example: $\mathfrak{g} = \mathfrak{sl}_3$

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & a & b \\ 1 & 0 & a \\ 0 & 1 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}.$$

Regular centralizers

Definition

The universal centralizer of $\mathfrak{g}$ is the smooth, symplectic algebraic variety

$$\begin{array}{c} \mathcal{Z} = \{(a, x) \in G \times \mathcal{S} \mid a \in G^x\} \\ \downarrow \\ \mathcal{S}. \end{array}$$

- Where does the symplectic structure come from?
- How to compactify the fibers of $\mathcal{Z}$?

  $\rightsquigarrow \overline{\mathcal{Z}} := \{(a, x) \in \overline{G} \times \mathcal{S} \mid a \in \overline{G^x}\}$
- How to extend the symplectic structure to $\overline{\mathcal{Z}}$?

Poisson spaces

Definition

A Poisson structure on a manifold M is a skew-symmetric, bilinear bracket

$$\{\cdot,\cdot\} : \mathcal{O}(M) \times \mathcal{O}(M) \longrightarrow \mathcal{O}(M)$$

which satisfies

- (Leibniz) $\{fg, h\} = f\{g, h\} + \{f, h\}g$
- (Jacobi) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

$$\begin{aligned} \text{Leibniz} \quad &\Rightarrow \quad \{\cdot,\cdot\} \text{ is a derivation in each coordinate} \\ &\Rightarrow \quad \exists\, \pi \in \Gamma(\wedge^2 T_M) \text{ such that } \{f, g\} = \pi(df, dg). \end{aligned}$$

Poisson spaces

The Poisson structure $\pi \in \Gamma(\wedge^2 T_M)$ is nondegenerate if

$$\begin{aligned}\pi^\# : T_M^* &\longrightarrow T_M \\ \alpha &\longmapsto \pi(\alpha, -)\end{aligned} \qquad \text{is an isomorphism.}$$

$\Leftrightarrow \exists$ a compatible symplectic $\omega \in \Gamma(\wedge^2 T_M^*)$.

More generally, the distribution $\pi^\#(T_M^*) \subset T_M$ is integrable—

$$M = \sqcup L.$$

$\rightsquigarrow \pi^\#$ descends to an isomorphism $T_L^* \longrightarrow T_L$

$\Rightarrow$ L is a symplectic manifold

$\Rightarrow$ M is foliated by symplectic leaves.

Example

Semisimple Lie algebra $\mathfrak{g}$

$\rightsquigarrow$ Kirillov–Kostant Poisson structure on $\mathfrak{g}^*$ ($\cong \mathfrak{g}$)

$$\{f,g\}_{\mathfrak{g}}(\xi) := \langle \xi, [d_\xi f, d_\xi g] \rangle$$

for $f, g \in \mathcal{O}(\mathfrak{g}^*)$, $\xi \in \mathfrak{g}^*$.

The symplectic leaves are the (co)adjoint orbits

$\Rightarrow$ (co)adjoint orbits are even-dimensional.

$\mathfrak{sl}_2\mathbb{R} \cong \mathfrak{sl}_2^*\mathbb{R}$:

(arXiv:1911.11732)

Poisson spaces

Definition

A Poisson transversal is a submanifold $X \subset M$ such that, for each symplectic leaf (L, ω),

- $X \pitchfork L$, and
- $\omega_{|X \cap L}$ is symplectic.

$\rightsquigarrow$ there is an induced Poisson structure $\pi_X \in \Gamma(\wedge^2 T_X)$ whose symplectic leaves are $(X \cap L, \omega_{|X \cap L})$.

Example

If (M, ω) is symplectic,

Poisson transversals $\leftrightsquigarrow$ symplectic submanifolds.

Poisson spaces

Example

The Kostant slice $\mathcal{S} \subset \mathfrak{g}$ is a Poisson transversal for the Kirillov–Kostant Poisson structure.

$$\begin{array}{ccc} G \times G \circlearrowright T^*_G \cong G \times \mathfrak{g} & & (a, x) \\ \downarrow \mu & & \downarrow \\ \mathfrak{g} \times \mathfrak{g} & & (\mathrm{Ad}_a x, x) \end{array}$$

μ is a Poisson map with image $\{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid x \in G \cdot y\}$

$$\begin{aligned} \Rightarrow \quad \mu^{-1}(\mathcal{S} \times \mathcal{S}) &= \mu^{-1}(\mathcal{S}_\Delta) \\ &= \{(a, x) \in G \times \mathcal{S} \mid \mathrm{Ad}_a x = x\} = \mathcal{Z}. \end{aligned}$$

$\Rightarrow$ the universal centralizer $\mathcal{Z}$ is a Poisson transversal (=symplectic submanifold) in T^*_G.

Compactified centralizers

$T^*_{\overline{G},D}$ logarithmic cotangent bundle of $\overline{G}$

- sections are logarithmic differential forms with poles along D
- canonical log-symplectic Poisson structure
  - top wedge power of Poisson bivector vanishes with minimal multiplicity on D
  - corresponds to a closed, nondegenerate logarithmic 2-form
  - open dense symplectic leaf is $T^*_G \subset T^*_{\overline{G},D}$
- fits into a diagram

$$\begin{array}{ccccc} T^*_G & \hookrightarrow & T^*_{\overline{G},D} & \hookrightarrow & \overline{G} \times \mathfrak{g} \times \mathfrak{g} \\ & \searrow & & \searrow & \downarrow \\ & & G & \hookrightarrow & \overline{G}. \end{array}$$

Compactified centralizers

$$\begin{array}{ccccc} G \times G & \circlearrowright & T^*_{\overline{G},D} & \hookleftarrow & T^*_G \\ & & \downarrow{\scriptstyle \overline{\mu}} & \swarrow{\scriptstyle \mu} & \\ & & \mathfrak{g} \times \mathfrak{g} & & \end{array}$$

$\overline{\mu}$ is a Poisson map with image $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$.

$$\Rightarrow \overline{\mu}^{-1}(\mathcal{S} \times \mathcal{S}) = \overline{\mu}^{-1}(\mathcal{S}_\Delta) \text{ is a Poisson transversal in } T^*_{\overline{G},D}.$$

Theorem [B.]

$$\overline{\mathcal{Z}} := \{(a, x) \in \overline{G} \times \mathcal{S} \mid a \in \overline{G^x}\}$$

is a smooth, log-symplectic algebraic variety which sits inside $T^*_{\overline{G},D}$ as the transversal $\overline{\mu}^{-1}(\mathcal{S}_\Delta)$.

Compactified centralizers

$\overline{G} = \sqcup \mathcal{O}_J \rightsquigarrow \overline{\mathcal{Z}} = \sqcup \mathcal{Z}_J$, where

$$\mathcal{Z}_J = \left\{(a, x) \in \overline{G} \times \mathcal{S} \mid a \in \overline{G^x} \cap \mathcal{O}_J\right\}.$$

Each orbit $\mathcal{O}_J$ is a fibration $\mathcal{O}_J \longrightarrow G/P_J \times G/P_J^-$

$$a \longmapsto (\mathfrak{p}_a, \mathfrak{p}_a^-),$$

and there is a canonical map

$$\mathfrak{c}_a : \mathfrak{p}_a \longrightarrow \mathfrak{p}_J/[\mathfrak{p}_J, \mathfrak{p}_J].$$

Theorem (cont.) [B.]

The symplectic leaves in the stratum $\mathcal{Z}_J$ are the fibers of the morphism

$$\begin{aligned} \mathcal{Z}_J &\longrightarrow \mathfrak{p}_J/[\mathfrak{p}_J, \mathfrak{p}_J] \\ (a, x) &\longmapsto \mathfrak{c}_a(x). \end{aligned}$$

Compactified centralizers

Let $H = (\sum \mathfrak{g}_{-\alpha_j}) \oplus \mathfrak{b}$

$$\rightsquigarrow \mu_H : G \times_B H \longrightarrow \mathfrak{g}$$
$$[g : y] \longmapsto \mathrm{Ad}_g\, y$$

The Hessenberg variety associated to $x \in \mathfrak{g}$ is

$$\begin{aligned}\mathrm{Hess}(x) &= \mu_H^{-1}(x) \\ &= \{gB \in G/B \mid \mathrm{Ad}_{g^{-1}}\, x \in H\}.\end{aligned}$$

Theorem (cont.) [B.]

There is an isomorphism of varieties over $\mathcal{S}$

$$\overline{\mathcal{Z}} \cong \{(gB, x) \in G/B \times \mathcal{S} \mid gB \in \mathrm{Hess}(x)\}.$$

In particular, for every $x \in \mathfrak{g}^{\mathrm{r}}$, $\overline{G^x} \cong \mathrm{Hess}(x)$.

Multiplicative analogues

Goal: $G \circlearrowright \mathfrak{g} \quad \rightsquigarrow \quad G \circlearrowright \tilde{G}$

Theorem [Steinberg]

There is an affine space

$$\Sigma \subset \tilde{G}^{\mathrm{r}},$$

of dimension l, which meets each regular conjugacy class in $\tilde{G}$ exactly once and transversally.

Example: $\tilde{G} = SL_4$

$$\Sigma = \left\{ \begin{pmatrix} a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}.$$

Multiplicative analogues

Definition

The (multiplicative) universal centralizer is the smooth, symplectic algebraic variety

$$\begin{array}{c} \mathfrak{Z} = \left\{(a, h) \in G \times \Sigma \mid aha^{-1} = h\right\}. \\ \downarrow \\ \Sigma \end{array}$$

$$\mathfrak{g} \rightsquigarrow \tilde{G}$$

$$\mathcal{S} \rightsquigarrow \Sigma$$

$$\mathcal{Z} \rightsquigarrow \mathfrak{Z}$$

$$T^*_G,\ T^*_{\overline{G},D} \rightsquigarrow ??$$

Multiplicative analogues

Definition [Alexeev–Kosmann-Schwarzbach–Meinrenken]

A quasi-Poisson structure on a $\tilde{G}$-manifold M is an invariant, skew-symmetric, bilinear bracket

$$\{\cdot,\cdot\} : \mathcal{O}(M) \times \mathcal{O}(M) \longrightarrow \mathcal{O}(M)$$

which satisfies

- (Leibniz) $\{fg, h\} = f\{g, h\} + \{f, h\}g$
- (twisted Jacobi) $\{f, \{g, h\}\} + c.p. = \chi_M(f, g, h)$

- $\exists\, \pi \in \Gamma(\wedge^2 T_M)$ such that $\{f, g\} = \pi(df, dg)$
- π is nondegenerate if there is a compatible quasi-Hamiltonian 2-form $\omega \in \Gamma(\wedge^2 T_M^*)$
- More generally, M is foliated by nondegenerate leaves.

Multiplicative analogues

Example

$\tilde{G} \circlearrowright \tilde{G}$ by conjugation $\rightsquigarrow$ qPoisson bivector $\pi \in \Gamma(\wedge^2 T_{\tilde{G}})^{\tilde{G}}$

nondegenerate leaves $\leftrightarrow$ conjugacy classes

Example

The double $\mathbb{D}_G := G \times \tilde{G}$ has a nondegenerate qPoisson structure relative to the $\tilde{G} \times \tilde{G}$-action

$$(g_1, g_2) \cdot (a, h) = (g_1 a g_2^{-1}, g_2 h g_2^{-1}).$$

$\rightsquigarrow$ group-valued moment map

$$\begin{aligned} \tilde{G} \times \tilde{G} \circlearrowright \ \mathbb{D}_G &\longrightarrow \tilde{G} \times \tilde{G} \\ (a, h) &\longmapsto (aha^{-1}, h) \end{aligned}$$

Multiplicative analogues

Theorem [B.]

Let (M, π) be a qPoisson $\tilde{G}$-manifold,

$M = \sqcup L$ its stratification by nondegenerate leaves,

with moment map

$$\Phi : M \longrightarrow \tilde{G}.$$

The Steinberg slice $M_\Sigma := \Phi^{-1}(\Sigma)$ is a submanifold of M,

with an induced Poisson structure $\pi_\Sigma \in \Gamma(\wedge^2 T_{M_\Sigma})$,

whose symplectic leaves are $\{M_\Sigma \cap L\}$.

In particular, (M, π) nondegenerate $\Rightarrow$ (M_Σ, π_Σ) symplectic.

Multiplicative analogues

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} \circlearrowright & \mathbb{D}_G & (a, h) \\ & \Big\downarrow \mu & \Big\downarrow \\ & \tilde{G} \times \tilde{G} & (aha^{-1}, h) \end{array}$$

The image of μ is $\{(g, h) \in \tilde{G} \times \tilde{G} \mid g \in G \cdot h\}$

$$\Rightarrow \quad \mu^{-1}(\Sigma \times \Sigma) = \mu^{-1}(\Sigma_\Delta)$$
$$= \{(a, h) \in G \times \Sigma \mid aha^{-1} = h\} = \mathfrak{Z}.$$

$\Rightarrow$ the multiplicative universal centralizer $\mathfrak{Z}$ is a Steinberg slice (=symplectic submanifold) in $\mathbb{D}_G$.

Recall the inclusions

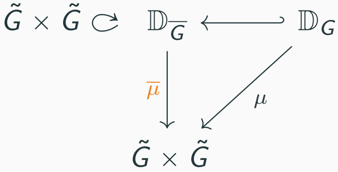
$$\begin{array}{ccccc} T^*_G & \hookrightarrow & T^*_{\overline{G},D} & \hookrightarrow & \overline{G} \times \mathfrak{g} \times \mathfrak{g} \\ & \searrow & & \searrow & \downarrow \\ & & G & \hookrightarrow & \overline{G}. \end{array}$$

Proposition [B.]

$T^*_{\overline{G},D}$ integrates to a smooth bundle of groups $\mathbb{D}_{\overline{G}}$, which fits into the diagram

$$\begin{array}{ccccc} \mathbb{D}_G & \hookrightarrow & \mathbb{D}_{\overline{G}} & \hookrightarrow & \overline{G} \times \tilde{G} \times \tilde{G} \\ & \searrow & & \searrow & \downarrow \\ & & G & \hookrightarrow & \overline{G}. \end{array}$$

Multiplicative analogues



$\overline{\mu}$ is a qPoisson moment map with image $\tilde{G} \times_{\tilde{G}//G} \tilde{G}$.

$$\Rightarrow \quad \overline{\mu}^{-1}(\Sigma \times \Sigma) = \overline{\mu}^{-1}(\Sigma_\Delta) \text{ is a Steinberg slice in } \mathbb{D}_{\overline{G}}.$$

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$$\overline{\mathfrak{Z}} := \{(a, h) \in \overline{G} \times \Sigma \mid a \in \overline{G^h}\}$$

is a smooth, log-symplectic algebraic variety which sits inside $\mathbb{D}_{\overline{G}}$ as the Steinberg slice $\overline{\mu}^{-1}(\mathcal{S}_\Delta)$.

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$$a \longmapsto (P_a, P_a^-),$$

and there is a canonical map

$$\mathfrak{c}_a : P_a \longrightarrow P_J/[P_J, P_J].$$

Theorem (cont.) [B.]

The symplectic leaves in the stratum $\mathfrak{Z}_J$ are the fibers of the morphism

$$\begin{aligned} \mathfrak{Z}_J &\longrightarrow P_J/[P_J, P_J] \\ (a, x) &\longmapsto \mathfrak{c}_a(x). \end{aligned}$$