Geometry of hybrid curves and their moduli spaces

with a view toward applications

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Stanford algebraic geometry seminar

March 11th 2022

Based on joint works with Noema Nicolussi.

- [AN20] Moduli of hybrid curves I: Variations of canonical measures
- [AN22] Moduli of hybrid curves II: Tropical and hybrid Laplacians
- [AN22b] Moduli of hybrid curves III: Algebraic geometry of hybrid curves

Motivation

Aim. Describe asymptotic geometry of Riemann surfaces

$\mathscr{M}_g$: moduli space of Riemann surfaces of genus g

$\bar{\mathscr{M}}_g$: Deligne-Mumford compactification

$\partial\bar{\mathscr{M}}_g = \bar{\mathscr{M}}_g \setminus \mathscr{M}_g$ the Deligne-Mumford boundary.

1. Limit of the Laplace operator $\Delta = \frac{1}{\pi i}\partial\bar{\partial}$
2. Limit of the canonical measures
3. Asymptotics of Green functions
4. Spectral convergence
5. ...

of Riemann surfaces when they approach a boundary point in $\bar{\mathscr{M}}_g$.

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Canonical measure on Riemann surfaces

S Riemann surface of genus g

μ_{Ar} Arakelov-Bergman measure on S

$$\mu_{\mathrm{Ar}} := \frac{i}{2}\sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j$$

$\omega_1, \dots, \omega_g$ orthonormal basis of $H^0(S, \omega_S)$ for

$$\langle \alpha, \beta \rangle := \frac{i}{2}\int_S \alpha \wedge \bar{\beta}$$

μ_{Ar}: positive density measure of total mass g.

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$\frac{1}{g}\mu_{\mathrm{Ar}}$: positive density measure of total mass one.

This is called the canonical measure of S denoted by μ^{can}.

Arakelov Green function

S canonically measured

Arakelov Green function

$$\mathrm{g}_S : \quad S \times S - \mathrm{diag}_S \quad \longrightarrow \quad \mathbb{R}$$

is the unique solution to

$$\frac{1}{\pi i}\partial_z \partial_{\bar{z}}\, \mathrm{g}(q, \cdot) = \delta_q - \mu^{can}, \qquad \int_S \mathrm{g}(q, y)\, d\mu^{can}(y) = 0$$

valid for all $q \in S$.

diag_S: diagonal embedding of $S \hookrightarrow S \times S$.

Appears in Arakelov geometry, String theory

Links to other invariants of Riemann surfaces: Faltings' δ-invariant, Modular graph functions (Green, d'Hoker, Piolin, Vanhove...), Zhang–Kawazumi invariant, ...

Variations of canonical measures and Green functions

$S_1, S_2, S_3, \ldots$ a sequence of Riemann surfaces of genus g

$s_1, s_2, s_3, \ldots$ corresponding points in $\mathscr{M}_g$

μ_j^{can}: canonical measure of S_j

g_j: Arakelov Green function of S_j

Question

Assume points $s_1, s_2, \ldots$ of $\mathscr{M}_g$ converge to a point s_∞ in $\overline{\mathscr{M}}_g$.

1. What is the limit of the sequence of measures $\mu_1^{can}, \mu_2^{can}, \ldots$?
2. Behavior of the Green functions $\mathrm{g}_1, \mathrm{g}_2, \ldots$ close to the boundary?

It turns out that

Deligne-Mumford compactification is not a suitable compactification.

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Hybrid compactification of $\mathscr{M}_g$

$\mathscr{M}_g^{hyb}$: *moduli space of hybrid curves of genus* g introduced in [AN20].

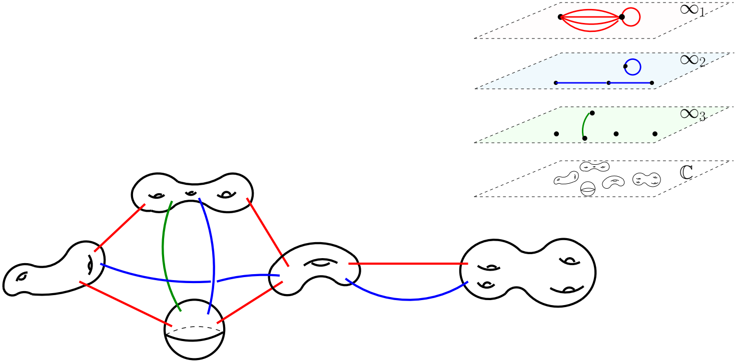

Figure: Example of a hybrid curve with the corresponding infinitary and finitary worldsheets.

In order to describe asymptotic geometry of Riemann surfaces, we need to

- Develop a hybrid function theory.

  Hybrid = Complex + Higher rank non-Archimedean, tropical

- Define hybrid canonical measures.
- Formulate a hybrid Poisson equation.
- Define hybrid Green functions.
- Use the formalism + link to asymptotic Hodge theory.

Moreover,

- Establish that algebraic geometry of Riemann surfaces survives in the hybrid limit.

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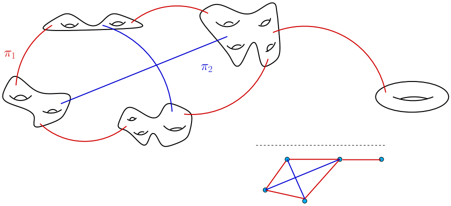
Objective of the talk

- Define hybrid curves and their moduli spaces
- Introduce hybrid function theory
- Algebraic geometry of hybrid curves
- Analytic geometry of hybrid curves
  (hybrid Poisson equation, hybrid canonical mesures, hybrid Green functions)
- Applications

Hybrid curves



Data underlying the definition

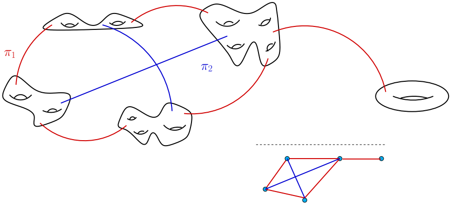
- Stable Riemann surface S with dual graph $G = (V, E)$
- Partition $\pi = (\pi_1, \dots, \pi_r)$ of the edge set E.
- Edge length function $l \colon E \to \mathbb{R}_{>0}$ with $l_j = l|_{\pi_j}$.

Metric realization

Obtained from (S, π, l) by plugging an a closed interval of length l_e instead of each node p_e on S.

This is called layered metrized curve complex.

Hybrid curves



Conformal equivalence at infinity and uniformization

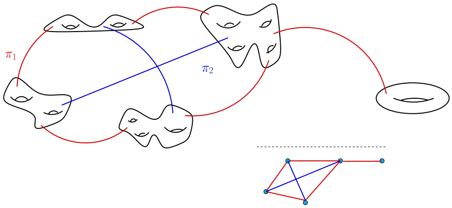
- $(S, \pi, l') \sim (S, \pi, l)$ if there exist $\lambda_1, \dots, \lambda_r > 0$ with $l'_j = \lambda_j l_j$.

Definition (Hybrid curve)

A hybrid curve $\mathscr{C}$ is the metric realization of a triple (S, π, l) up to conformal equivalence at infinity.

The integer r is called the rank of $\mathscr{C}$. The underlying conformal equivalence class of layered metric graphs is called a tropical curve. There is a natural representatives $\mathcal{M}\mathscr{C}$ and Γ in each class by requiring $\sum_{e \in \pi_j} l_j(e) = 1$. (normalization property)

Hybrid curves



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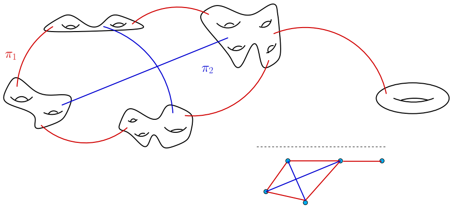

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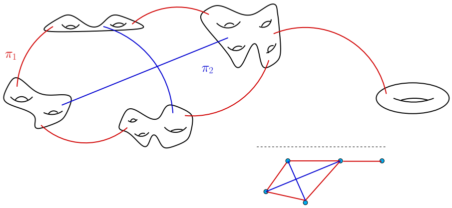

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Infinitary and finitary layers

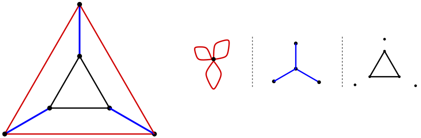


Figure: A tropical curve of rank three with underlying ordered partition $\pi = (\pi_1, \pi_2, \pi_3)$, with three infinitary layers $\Gamma^1, \Gamma^2, \Gamma^3$.

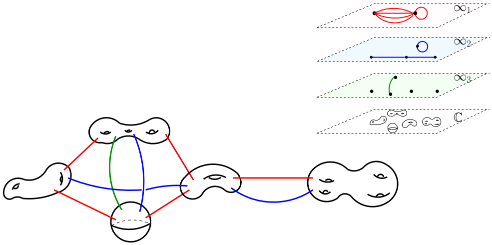


Figure: A hybrid curve $\mathscr{C}$ of rank three with the corresponding infinitary layers $\Gamma^1, \Gamma^2, \Gamma^3$, and finitary (complex) layer $\pi_{\mathbb{C}}$.

Hybrid curves arise naturally in the study of multiparameter families of complex curves. The situation to have in mind:

$\mathcal{X}/B$: Generically smooth family of complex curves

$D \subset B$ discriminant locus, simple normal crossing (after a base change to a log-resolution).

Hybrid curves replace singular fibers of the family and give rise to a hybrid family $\mathcal{X}^{hyb}$ defined on the hybrid replacement B^{hyb} of the pair (B, D).

B^{hyb}: higher rank hybrid compactification of $B \setminus D$ defined in [AN20].

(Refinement of the compactification of Boucksom-Jonsson, Berkovich.)

(Higher rank) tropical and hybrid compactifications

$\Delta = \{z \mid |z| \leq 1\}$ closed disk of radius one.

$\Delta^* = \{z \mid 0 < |z| \leq 1\}$.

Consider Δ^d and $U = \Delta^{*r} \times \Delta^{d-r}$.

Locally (B, D) is of the form $(\Delta^d, \Delta^d \setminus U)$.

Enough to treat the case $B = \Delta^d$ and $D = \Delta^d \setminus U$.

$$\begin{array}{rcl} \mathrm{Log}\colon U & \longrightarrow & \eta := \mathbb{R}_+^r \\ (z_1, \dots, z_d) & \mapsto & (-\log|z_1|, \dots, -\log|z_r|). \end{array}$$

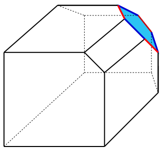
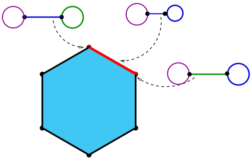


Figure: Tropical compactification of a cone and its part of full sedentarity.



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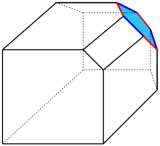

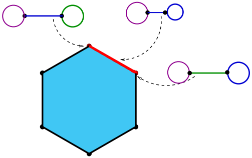

$\overline{\eta}^{\,trop}$ the tropical compactification of η.

$B^{hyb} :=$ topological closure of $U \hookrightarrow B \times \overline{\eta}^{\,trop}$

B^{hyb} is well-defined, that is, independent of choice of coordinates.

Definition (Hybrid moduli space)

Hybrid moduli space $\mathscr{M}_g^{hyb}$ is the hybrid compactification of $\mathscr{M}_g$ associated to the pair $(\overline{\mathscr{M}}_g, \partial\overline{\mathscr{M}}_g)$

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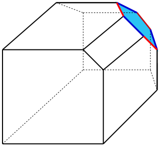

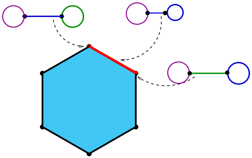

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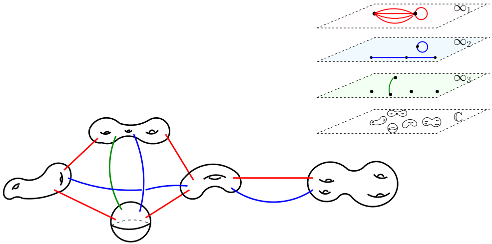
Meromorphic functions on hybrid curves



$\mathscr{C}$: hybrid curve with underlying normalized triple $(S, \pi = (\pi_1, \dots, \pi_r), l)$,
$\pi_{\mathbb{C}} := \widetilde{S}$ normalization of S

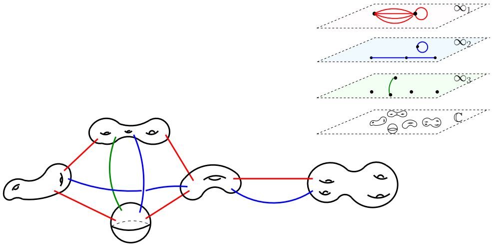
A hybrid meromorphic function on $\mathscr{C}$ is an $(r+1)$-tuple

$$\boldsymbol{f} = (f_1, \dots, f_r, f_{\mathbb{C}}), \qquad \text{consisting of functions}$$

- $f_j : \Gamma^j \to \mathbb{R}$ meromorphic on infinitary layers Γ^j, $j \in [r]$,

  meromorphic = continuous, piecewise affine with integral slopes.

Meromorphic functions on hybrid curves



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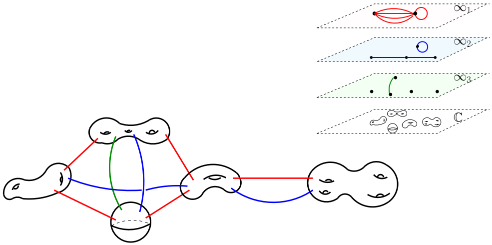

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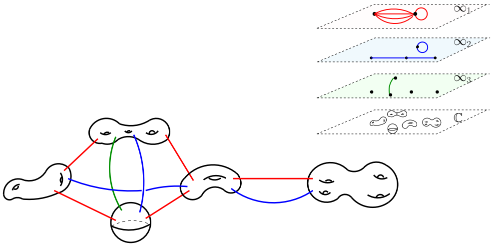
- $f_{\mathbb{C}} : \pi_{\mathbb{C}} \to \mathbb{C}$ meromorphic on complex part $\pi_{\mathbb{C}}$.

Divisor theory on hybrid curves

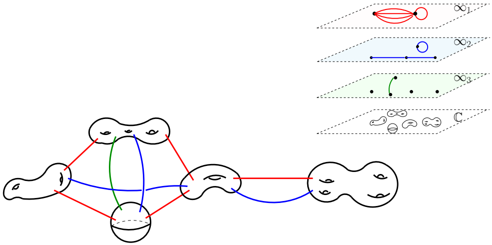


$\mathscr{C}$: hybrid curve

$\mathrm{Div}(\mathscr{C})$: free abelian group generated by points of $\mathscr{C}$.

$$D = \sum_{x \in \mathscr{C}} D(x)\, x.$$

Divisor theory on hybrid curves



Divisor of a meromorphic function $\boldsymbol{f}$:

$$\mathrm{div}(\boldsymbol{f}) = \sum_{x \in \mathscr{C}} \mathrm{ord}_x(\boldsymbol{f}) x.$$

order of vanishing at x defined by

- x in the middle of e, $e \in \pi_j$, then $\mathrm{ord}_x(\boldsymbol{f})$ sum of slopes of f_j along two unit tangents at x on e.
- x in the smooth part of S, then $\mathrm{ord}_x(\boldsymbol{f}) = \mathrm{ord}_x(f_{\mathbb{C}})$.
- x point of attachment of e to C_v, $e \in \pi_j$, then $\mathrm{ord}_x(\boldsymbol{f})$ is sum of $\mathrm{ord}_x(f_{\mathbb{C}})$ and slope of f_j along tangent vector at x on e.

Hybrid Riemann-Roch

Theorem (Hybrid Riemann-Roch AN22b)

For any divisor $D \in \mathrm{Div}(\mathscr{C})$,

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

K: canonical divisor of $\mathscr{C}$

$$K = \sum_{v \in V} (\omega_{C_v} + A_v)$$

ω_{C_v} canonical divisor of C_v

$A_v = \sum_{e \sim v} p_v^e$ divisor on C_v

Generalizes graph (Baker-Norine), metric graph (Gahtman-Kerber, Mikhalkin-Zharkov), metrized curve complex (A.-Baker) Riemann-Roch theorems

Hybrid Abel-Jacobi theorem

Theorem (Hybrid Abel-Jacobi AN22-b)

There is an Abel-Jacobi map

$$\mathrm{AJ}_{\mathscr{C}} \colon \mathrm{Pic}^0(\mathscr{C}) \longrightarrow \Omega^1(\mathscr{C})^* / H_1(\mathscr{C}).$$

The hybrid Abel-Jacobi map AJ *is moreover an isomorphism.*

$\mathrm{Pic}^0(\mathscr{C})$ and $\Omega^1(\mathscr{C})$ come with natural filtrations.

$\Omega^1(\mathscr{C})^* := \mathrm{Hom}\Big(\Omega^1(\mathscr{C}), \mathbb{R}^r \times \mathbb{C}\Big)$ which respects the twofiltration.

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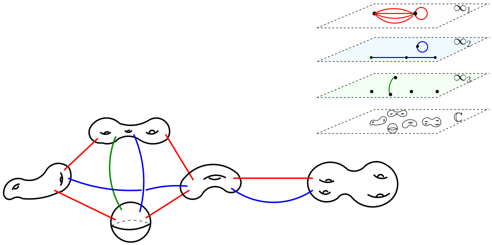
Function theory on hybrid curves



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A (complex-, real-valued) hybrid function on $\mathscr{C}$ is an $(r+1)$-tuple

$$\boldsymbol{f} = (f_1, \dots, f_r, f_{\mathbb{C}}), \qquad \text{consisting of functions}$$

- $f_j : \Gamma^j \to \mathbb{C}, \mathbb{R}$ on infinitary layers Γ^j, $j \in [r]$, and
- $f_{\mathbb{C}} : \pi_{\mathbb{C}} \to \mathbb{C}, \mathbb{R}$ on the complex part $\pi_{\mathbb{C}}$.

$\boldsymbol{f} = (f_1, \ldots, f_r, f_{\mathbb{C}})$ on $\mathscr{C}$ is called *continuous*, *smooth*, etc., if f_j are continuous, smooth, etc., respectively.

The infinitary parts f_j, $j = 1, \ldots, r$, are supposed to be continuous and piecewise smooth.

Each f_j gives rise to pull-back function $f_j^* \colon \Gamma \to \mathbb{C}, \mathbb{R}$ by linear interpolation.

The finitary (complex) part is supposed to be smooth, or, more generally, to have finitely many logarithmic poles on $\pi_{\mathbb{C}}$.

The pull-back $f_{\mathbb{C}}^* \colon \mathscr{MC} \to \mathbb{C}, \mathbb{R}$ is well-defined if $f_{\mathbb{C}}$ has no logarithmic poles.

Otherwise, we first regularize $f_{\mathbb{C}}$ to $f_{\mathbb{C}}^{\mathrm{reg}}$ and then take the linear interpolation.

Regularization depends on the choice of a local coordinate.

$$f_{\mathbb{C}}^{\mathrm{reg}}(p_v^e) := \lim_{p \to p_v^e} f_{\mathbb{C}}(p) - r_v^e \log |z_v^e|,$$

for local parameter z_v^e around p_v^e.

In practice, this is given by an *adapted system of coordinates*, the existence of which is a consequence of Hubbard-Koch '14 analytic construction of $\overline{\mathscr{M}}_g$.

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Otherwise, we first regularize $f_{\mathbb{C}}$ to $f_{\mathbb{C}}^{\mathrm{reg}}$ and then take the linear interpolation.

Regularization depends on the choice of a local coordinate.

$$f_{\mathbb{C}}^{\mathrm{reg}}(p_v^e) := \lim_{p \to p_v^e} f_{\mathbb{C}}(p) - r_v^e \log |z_v^e|,$$

for local parameter z_v^e around p_v^e.

In practice, this is given by an *adapted system of coordinates*, the existence of which is a consequence of Hubbard-Koch '14 analytic construction of $\overline{\mathscr{M}}_g$.

$\boldsymbol{f} = (f_1, \ldots, f_r, f_{\mathbb{C}})$ on $\mathscr{C}$ is called *continuous*, *smooth*, etc., if f_j are continuous, smooth, etc., respectively.

The infinitary parts f_j, $j = 1, \ldots, r$, are supposed to be continuous and piecewise smooth.

Each f_j gives rise to pull-back function $f_j^* \colon \Gamma \to \mathbb{C}, \mathbb{R}$ by linear interpolation.

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Layered measures

A layered measure $\boldsymbol{\mu}$ on $\mathscr{C}$ is an $r+1$-tuple $(\mu_1, \dots, \mu_r, \mu_{\mathbb{C}})$ of measures μ_j on Γ^j, $j = 1, \dots, r$, and $\mu_{\mathbb{C}}$ on $\pi_{\mathbb{C}}$.

We define the mass function $\mathscr{M}ass$ defined on the connected components of the layers by

$$\mathscr{M}ass(H) := \mu_j(H) - \mu_{j-1}(\{x_H\}),$$

where x_H is the point of Γ^{j-1} associated to the contraction of H.

$\boldsymbol{\mu}$ is called of mass zero if $\mathscr{M}ass$ vanishes.

A layered measure $\boldsymbol{\nu}$ is called of mass one if $\mathscr{M}ass(\Gamma^1) = 1$ and outside Γ^1, $\mathscr{M}ass$ vanishes.

Hybrid Laplacian

$\mathscr{C}$: hybrid curve of rank r

$\boldsymbol{f} = (f_1, \dots, f_r, f_{\mathbb{C}})$ hybrid function

$$\boldsymbol{\Delta}\colon \text{Hybrid functions} \longrightarrow \text{Layered measures of mass zero}$$

$$\boldsymbol{\Delta}(\boldsymbol{f}) = \boldsymbol{\Delta}_1(f_1) + \boldsymbol{\Delta}_2(f_2) + \dots \boldsymbol{\Delta}_r(f_r) + \boldsymbol{\Delta}_{\mathbb{C}}(f_{\mathbb{C}})$$

$$\boldsymbol{\Delta}_j(f_j) := \Big(0, \dots, 0, \Delta_j(f_j), \mathrm{div}_{j \prec j+1}(f_j), \dots, \mathrm{div}_{j \prec r}(f_j), \mathrm{div}_{j \prec \mathbb{C}}(f_j)\Big).$$

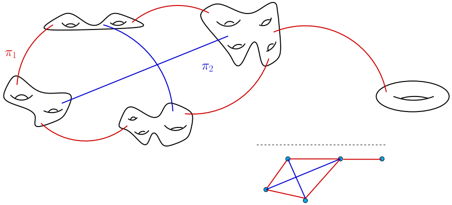
$$\boldsymbol{\Delta}_{\mathbb{C}}(f_{\mathbb{C}}) := \Big(0, \dots, 0, \Delta_{\mathbb{C}}(f_{\mathbb{C}})\Big)$$

with

Δ_j: metric graph Laplacian on Γ^j (Kirchhoff Laplacian)

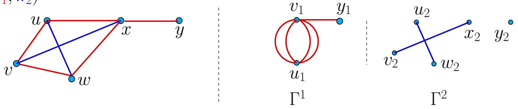
$\Delta_{\mathbb{C}}$: $\frac{1}{\pi i}\partial\bar{\partial}$ Laplacian on Riemann surfaces

$$\mathrm{div}_{j \prec i}(f_j) := -\sum_{e \in \pi_j} \sum_{v \in e} \mathrm{sl}_e f_j(v)\, \delta_{\mathrm{p}_j(v)},$$



Γ

$\pi = (\pi_1, \pi_2)$



$\boldsymbol{f}^{trop} = (f_1, f_2)$

$f_1(u_1) = a,\ f_1(v_1) = b,\ f_1(y_1) = c$

Assume f_1 affine linear on each red edge. Then,

$$\begin{aligned} \mathrm{div}_{1 \prec 2}(f_1) = {} & 10(a-b)\delta_{u_2} + 10(a-b)\delta_{w_2} + 10(b-a)\delta_{v_2} \\ & + (10(b-a) + 5(b-c))\delta_{x_2} + 5(c-b)\delta_{y_2}. \end{aligned}$$

Theorem (AN22)

The hybrid Laplace operator is a weak limit of the Laplace operator on Riemann surfaces.

That is, using an appropriate notion of logarithm map from Riemann surfaces to hybrid curves, we pull-back $\boldsymbol{f}$ to nearby Riemann surfaces to get functions f_t, $t \in \mathscr{M}_g$.

Then, we have for any continuous function on the universal family $\mathscr{C}_g^{hyb}$,

$$\int_{\mathscr{C}_t} h|_{\mathscr{C}_t} \Delta f_t \to \int_{\mathscr{C}_{\mathbf{t}}} h|_{\mathscr{C}_{\mathbf{t}}} \boldsymbol{\Delta f}$$

as t tends to $\mathbf{t}$ in $\mathscr{M}_g^{hyb}$.

Hybrid Poisson equation I

$\mathscr{C}$: hybrid curve of rank r

$\boldsymbol{\mu} = (\mu_1, \dots, \mu_r, \mu_{\mathbb{C}})$: layered measure of mass zero on $\mathscr{C}$

Consider the equation on $\mathscr{C}$:

$$\boldsymbol{\Delta f} = \boldsymbol{\mu}$$

This is a coupled system of Poisson equations

$$\begin{cases} \Delta_1(f_1) = \mu_1 & \text{on } \Gamma^1 \\ \Delta_2(f_2) = \mu_2 - \mathrm{div}_{1 \prec 2}(f_1) & \text{on } \Gamma^2 \\ \dots & \\ \Delta_{\mathbb{C}}(f_{\mathbb{C}}) = \mu_{\mathbb{C}} - \sum_{j=1}^r \mathrm{div}_{j \prec \mathbb{C}}(f_j) & \text{on } \pi_{\mathbb{C}} \end{cases}$$

on metric graphs Γ^j, $j \in [r]$, and $\pi_{\mathbb{C}}$.

Solutions exist. However,

Issue Dimension of the space of solutions = total number of connected components of layers.

Too many solutions!

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on metric graphs Γ^j, $j \in [r]$, and $\pi_{\mathbb{C}}$.

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Hybrid Poisson equation II

Uniqueness of solutions (modulo constants) will be guaranteed through a specific condition in the theory named

harmonically arranged property.

$\mathscr{C}$: hybrid curve of rank r

$\boldsymbol{\mu}$: layered measure of total mass zero

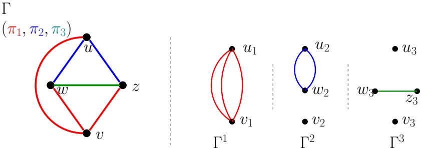
$\boldsymbol{\nu}$: layered measure of mass one.

Consider the following hybrid Poisson equation

$$\begin{cases} \boldsymbol{\Delta f} = \boldsymbol{\mu} \\ \boldsymbol{f} \text{ harmonically arranged} \\ \int_{\mathscr{C}} \boldsymbol{f}\, d\boldsymbol{\nu} = 0. \end{cases} \quad (1)$$

Theorem (Existence and uniqueness of solutions of the hybrid Poisson equations AN22)

The hybrid Poisson equation has a unique solution $\boldsymbol{f}$ for every bimeasured hybrid curve $(\mathscr{C}, \boldsymbol{\mu}, \boldsymbol{\nu})$.

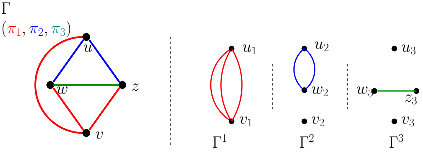


$\boldsymbol{f} = (f_1, f_2, f_3)$
$f_2(u_2) = a$, $f_2(v_2) = b$, $f_2(w_2) = c$
$f_3(u_3) = a'$, $f_3(v_3) = b'$, $f_3(w_3) = c'$, $f_3(z_3) = d'$

f_j is called *lower harmonic* if it is harmonic up to layer $j-1$. This means, the slopes of f_j on π_i for all $i < j$ satisfy the harmonicity property around all vertices w of Γ^i,

$$\sum_{u \in \mathrm{p}_i^{-1}(w)} \sum_{\substack{e=uv \\ e \in \pi_i}} \frac{f_j(v) - f_j(u)}{l_i(e)} = 0.$$

In the example, f_2 is lower harmonic if $b - a + 2(b-c) = 3b - a - 2c = 0$, that is, if $b = \frac{a+2c}{3}$.
f_3 is lower harmonic if $3b' = a' + b' + c'$ and $2a' = c' + d'$.



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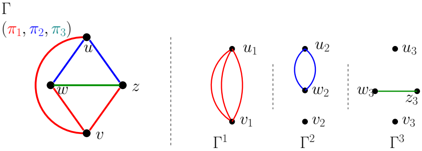

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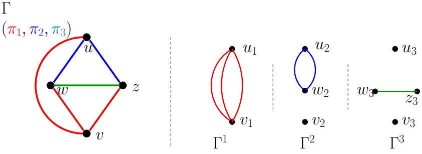

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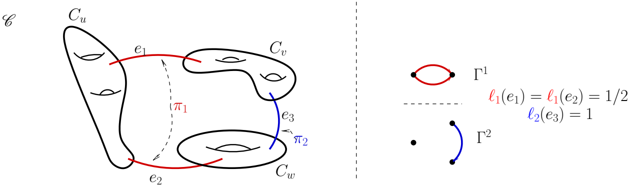
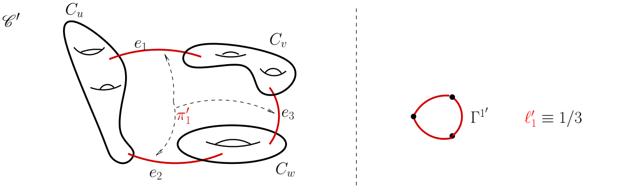
$\boldsymbol{f} = (f_1, \dots, f_r, f_{\mathbb{C}})$ is harmonically arranged if f_j is lower harmonic for $j = 1, \dots, r, \mathbb{C}$.

Hybrid canonical measures



$$6\mu_{\mathscr{C}} = \mu_{C_v} + \mu_{C_u} + \mu_{C_w} + d\theta_{e_1} + d\theta_{e_2}$$



$$6\mu_{\mathscr{C}'} = \mu_{C_v} + \mu_{C_u} + \mu_{C_w} + d\theta_{e_1} + d\theta_{e_2} + d\theta_{e_3}$$

Figure: Example of two hybrid curves with the same underlying stable Riemann surface. The canonical measures have the same Archimedean parts. The non-Archimedean parts are however different.

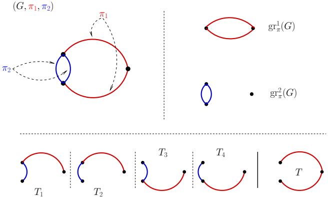


Figure: A tropical curve and its four spanning trees in $\mathcal{T}_\pi(G)$.

$$\mu^{can} := \frac{1}{g}\Big(\mu_{\mathrm{Ar},\mathbb{C}} + \sum_{e\in E} \ell_e^{-1}\mu(e)\, d\theta_e\Big),$$

- $d\theta_e$ uniform Lebesgue measure on the edge $e \in E$.
- $\mu(e)$ Foster coefficient defined by

$$\mu(e) = \frac{1}{\sum_{T\in\mathcal{T}(G)} \omega(T)} \sum_{T\in\mathcal{T}_\pi(G):\ e\notin E(T)} \omega(T). \tag{2}$$

- for spanning tree T of G, $\omega(T) = \prod_{e\in E\setminus E(T)} \ell_e$.
- $\mu_{\mathrm{Ar},\mathbb{C}}$ is the Arakelov-Bergman measure of the components.

Hybrid canonical measures

Hybrid moduli space $\mathscr{M}_g^{hyb}$ comes with its universal family $\mathscr{C}_g^{hyb}$ (defined on hybrid étale charts B^{hyb})

Theorem (AN20)

Canonically measured family $(\mathscr{C}_g^{hyb}, \mu^{can})$ is continuous over $\mathscr{M}_g^{hyb}$.

Continuity is in the distributional sense:

For any continuous $f : \mathscr{C}_g^{hyb} \to \mathbb{R}$, integration along fibers is continuous, i.e.,

$$F(\mathbf{t}) := \int_{\mathscr{C}_{\mathbf{t}}^{hyb}} f_{|_{\mathscr{C}_{\mathbf{t}}^{hyb}}} \, d\mu_{\mathbf{t}}^{can}, \qquad \mathbf{t} \in \mathscr{M}_g^{hyb}$$

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Question

Assume $s_1, s_2, \dots$ of $\mathscr{M}_g$ converge to a point s_∞ in $\overline{\mathscr{M}}_g$.

What is the limit of the sequence of measures $\mu_1^{can}, \mu_2^{can}, \dots$?

Answer. If $s_1, s_2, \dots$ converges to a point $\mathbf{t}$ in $\mathscr{M}_g^{hyb}$, then the limit is the measure $\mu_{\mathbf{t}}^{can}$ defined on $\mathscr{C}_{\mathbf{t}}^{hyb}$.

Otherwise, there is no limit. ($\mathscr{M}_g^{hyb}$ is Hausdorff.)

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Hybrid Green functions

$\mathscr{C}$: hybrid curve of rank r

$\boldsymbol{\mu} = (\mu_1, \dots, \mu_r, \mu_{\mathbb{C}})$: layered measure of total mass one

$\mu_{\mathbb{C}}$ continuous $(1,1)$-form.

Any point $p \in \mathscr{C}$ induces a layered Dirac measure $\boldsymbol{\delta}_p$ on $\mathscr{C}$.

The hybrid Poisson equation

$$\begin{cases} \boldsymbol{\Delta f} = \boldsymbol{\delta}_p - \boldsymbol{\mu}, \\ \boldsymbol{f} \text{ is harmonically arranged,} \\ \int_{\mathscr{C}} \boldsymbol{f}\, d\boldsymbol{\mu} = 0 \end{cases} \tag{3}$$

has a unique solution denoted by

$$\mathbf{g}_{\boldsymbol{\mu}}(p, \cdot) = \big(\mathrm{g}_{\boldsymbol{\mu},1}(p,\cdot), \dots, \mathrm{g}_{\boldsymbol{\mu},r}(p,\cdot), \mathrm{g}_{\boldsymbol{\mu},\mathbb{C}}(p,\cdot)\big)$$

called the hybrid Green function associated to $\boldsymbol{\mu}$.

Asymptotics of Arakelov Green functions

Theorem (Layered expansion of the Arakelov Green function AN22)

For $(t, p_t) \in \mathscr{M}_{g,1}$ converging to the hybrid limit $(\mathbf{t}, p_{\mathbf{t}}) \in \mathscr{M}_{g,1}^{hyb}$ corresponding to hybrid curve $\mathscr{C}$ with underlying triple (S, π, l), we can write uniformly

$$\mathrm{g}_t(p_t, \cdot) = L_1(t)\hat{\mathrm{g}}_{t,1}(p_t, \cdot) + L_2(t)\hat{\mathrm{g}}_{t,2}(p_t, \cdot) + \cdots + L_r(t)\hat{\mathrm{g}}_{t,r}(p_t, \cdot) + \hat{\mathrm{g}}_{t,\mathbb{C}}(p_t, \cdot) + o(1)$$

where

(1) *Functions $\hat{\mathrm{g}}_{t,j}(p_t, \cdot)$ converge to $\mathrm{g}_{\mathbf{t},j}(p_{\mathbf{t}}, \cdot)$, and*

(2) *The tuple $\mathbf{g}_{\mathbf{t}}(p_{\mathbf{t}}, \cdot) = (\mathrm{g}_{\mathbf{t},1}(p_{\mathbf{t}}, \cdot), \dots, \mathrm{g}_{\mathbf{t},r}(p_{\mathbf{t}}, \cdot), \mathrm{g}_{\mathbf{t},\mathbb{C}}(p_{\mathbf{t}}, \cdot))$ is the hybrid Green function associated to the canonical measure $\mu_{\mathbf{t}}^{can}$.*

Here $L_j(t) = -\sum_{e \in \pi_j} \log |z_e(t)|$ for analytic coordinates z_e around the point $s \in \overline{\mathscr{M}}_g$.

The tuple $\hat{\mathbf{g}}_{\mathbf{t}}(p_{\mathbf{t}}, \cdot) = (\hat{\mathrm{g}}_{\mathbf{t},1}(p_{\mathbf{t}}, \cdot), \dots, \hat{\mathrm{g}}_{\mathbf{t},r}(p_{\mathbf{t}}, \cdot), \hat{\mathrm{g}}_{\mathbf{t},\mathbb{C}}(p_{\mathbf{t}}, \cdot))$ is a hybrid Green function for the pushout to $\mathscr{C}$ of μ_t^{can} via an appropriate log map.

Comments

1. Theorem implies a similar statement for other families.
2. For a one-parameter family of Riemann surfaces $\mathcal{S}$ (defined over the punctured disk)
   - Wentworth '91. Refined asymptotics where the limit has a unique node.
   - de Jong '19. Dominant term.
3. For multiparameter families, Faltings '20.

   Indicated that the asymptotics is separated into two terms, graph and Riemann surface parts. Mentioned problematic issues in further description of these terms.