Resolutions of Richardson varieties, stable curves, and dual simplicial spheres

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Stanford Algebraic Geometry Seminar, January 2022

Abstract

To a simple normal crossings divisor (sncd) D, one associates its "dual simplicial complex", with a vertex for each component D_i and face F for each stratum $\cap_{f\in F} D_f \neq \emptyset$. For example, Escobar's brick manifolds (which resolve Richardson varieties) come with an sncd whose dual complex is a subword complex. In good cases (like hers) the dual complex is a sphere.

With no such geometrical input, Björner-Wachs showed that the order complex of a Bruhat interval (u, v) is a sphere. I'll define a space of equivariant stable maps from $\mathbb{P}^1$ to the Richardson variety X_u^v (making no choices, e.g. of reduced words), and prove that this space is a smooth orbifold, which comes with a natural sncd whose dual is the Björner-Wachs complex. In the Grassmannian case this space is GKM, and I describe its GKM graph in terms of rim-hook tableaux.

These slides are available at `http://pi.math.cornell.edu/~allenk/`

Simple normal crossing divisors and their dual complexes.

Let $D_1, D_2, \ldots, D_m$ be a collection of smooth divisors in a (complex, say) manifold M. They are **simple normal crossings** if $\bigcap_{f \in F} D_f$ is smooth connected of codimension $|F|$ (when nonempty) for each $F \subseteq [m]$, i.e. rather like a set of coördinate hyperplanes in $\mathbb{C}^n$. Their union $D = D_1 \cup \ldots \cup D_m$ is a **simple normal crossings divisor** or **sncd**.

A good test case is $M = TV_P$ the projective toric variety associated to a polytope P, and D the complement of the open T-orbit. Then $\cap_{f \in F} D_F$ is always irreducible (when nonempty), but will only have always the right codimension when M is orbifold, i.e. when P is "simple". Consider a pyramid for counterexamples.

Another nonexample is $M = \mathbb{CP}^2 = \{[x : y : z]\}$, $D_1 = \{x = 0\}$, $D_2 = \{y^2 = xz\}$. The intersection $D_1 \cap D_2$ is smooth and codim 2 but disconnected.

Yet another is the Schubert divisors in the 3-fold GL_3/B, two smooth surfaces whose intersection $\mathbb{P}^1 \cup_{pt} \mathbb{P}^1$ is not smooth.

When D *is* snc, define its **dual complex** $\Delta(D) \subseteq 2^{[m]}$ to be the simplicial complex with vertex set $[m]$, where $F \subseteq [m]$ to be a face iff $\bigcap_{f \in F} D_f \neq \emptyset$.

[Kollár '14] showed that *every* simplicial complex arises as the dual of some sncd – but states in [Kollár-Xu '16] a "folklore conjecture": if D is anticanonical in M, then $\Delta(D)$ is homeomorphic to a sphere mod a finite group.

Bott-Samelson manifolds and their boring sncds.

Fix a pinning (G, B, T, W) of a Lie (or Kac-Moody) group. Given a word Q in the simple reflections of the Weyl group W, define the **Bott-Samelson manifold**

$$BS^Q := \left\{ (F_0, \ldots, F_{\#Q}) \in (G/B)^{1+\#Q} : F_0 = B/B,\ \forall i\ (F_i, F_{i+1}) \in \overline{G_\Delta \cdot (B/B, r_{q_i} B/B)} \right\}$$

of tuples of (generalized) flags, starting at the base flag B/B and only changing a little bit at each step. This is an iterated $\mathbb{P}^1$ bundle, hence smooth projective irreducible, and possesses a B-action, with $(BS^Q)^T$ isolated and $\cong 2^Q$.

The **Bott-Samelson map** $BS^Q \to G/B$ takes $(F_i) \mapsto F_{\#Q}$, with image some B-orbit closure $X^w := \overline{BwB}/B$. This w is the **Demazure product** of Q, the (unique) maximum product of any subword of Q. (In the boring case for us $w = \prod Q$, though people like that $BS^Q \twoheadrightarrow X^w$ is then a resolution of singularities.)

Whenever $F_{i-1} = F_i$, we might as well skip letter i in Q, giving us an injection $BS^{Q \setminus i} \hookrightarrow BS^Q$. Intersecting these images we get a stratum $\cong BS^R$ for each of the $2^{\#Q}$ many subwords $R \subseteq Q$. Every intersection is nonempty!

Hence if $D = \bigcup_{i=1}^{\#Q} BS^{Q \text{ minus letter } i}$, it forms an sncd in BS^Q whose $\Delta(D)$ is the entire simplex, rather than some interesting subcomplex of that simplex.

Brick manifolds and spherical subword complexes.

The **brick manifold** $\mathtt{Brick}^Q \subseteq BS^Q$ is the $F_{\#Q} = wB/B$ fiber of $BS^Q \twoheadrightarrow X^w$ (w being the Demazure product). It is smooth (by Sard), T-invariant, and of dimension $\#Q - \ell(w)$ (so, boring when Q reduced).
Let $D = \bigcup_{q \in Q} (\mathtt{Brick}^Q \cap BS^{Q \setminus q}) \subseteq \mathtt{Brick}^Q$; it is an sncd in $\mathtt{Brick}^Q$.

Theorem [Escobar '16]. $\Delta(D)$ is the "subword complex" $\Delta(Q, w)$ whose facets are the complements $Q \setminus R$ of reduced subwords $R \subseteq Q$ with product w. It is therefore homeomorphic to a sphere [K-Miller '05].

Since D is anticanonical in $\mathtt{Brick}^Q$, this is consonant with the folklore conjecture.

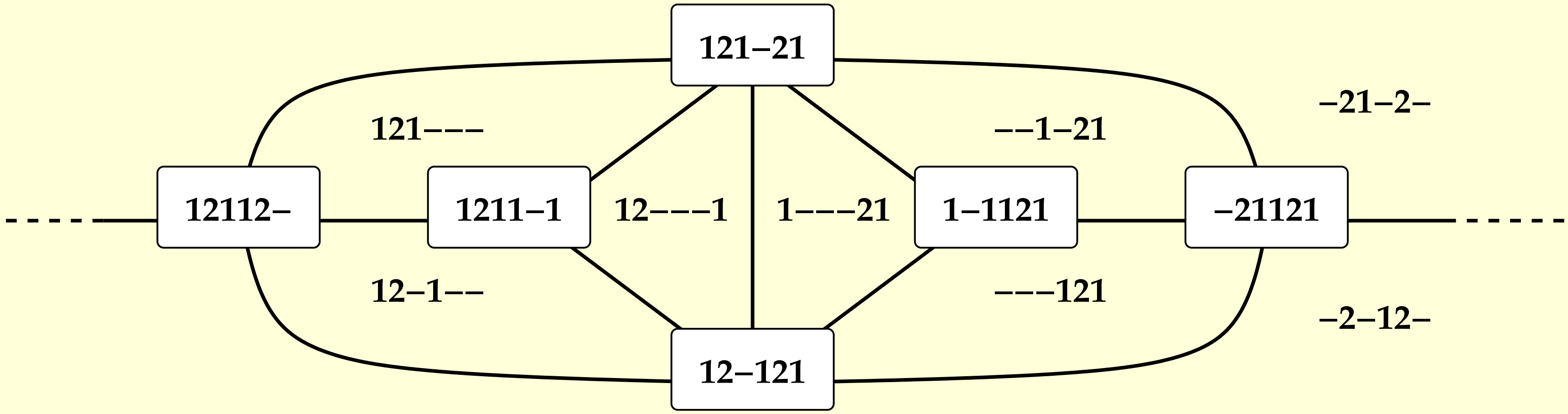
A **Richardson variety** $X_u^v \subset G/B$ is the transverse intersection of a Schubert variety $X_u := \overline{B_- u B}/B$ and an opposite Schubert variety $X^v := \overline{BvB}/B$.

We can resolve $X_u = w_0 X^{w_0 u}$ using $BS_R := w_0 BS^R$, where R is a reduced word for $w_0 u$. Brion constructed a resolution of X_u^v using the fiber product of $BS^Q \twoheadrightarrow X^v$ and $BS_R \twoheadrightarrow X_u$. This fiber product is naturally identified with the brick manifold $\mathtt{Brick}^{Q\overleftarrow{R}}$, where $\overleftarrow{R}$ is R reversed, and the map to G/B takes $(F_0, F_1, \ldots, F_{\#Q}, \ldots, F_{\#Q+\#R}) \mapsto F_{\#Q}$.

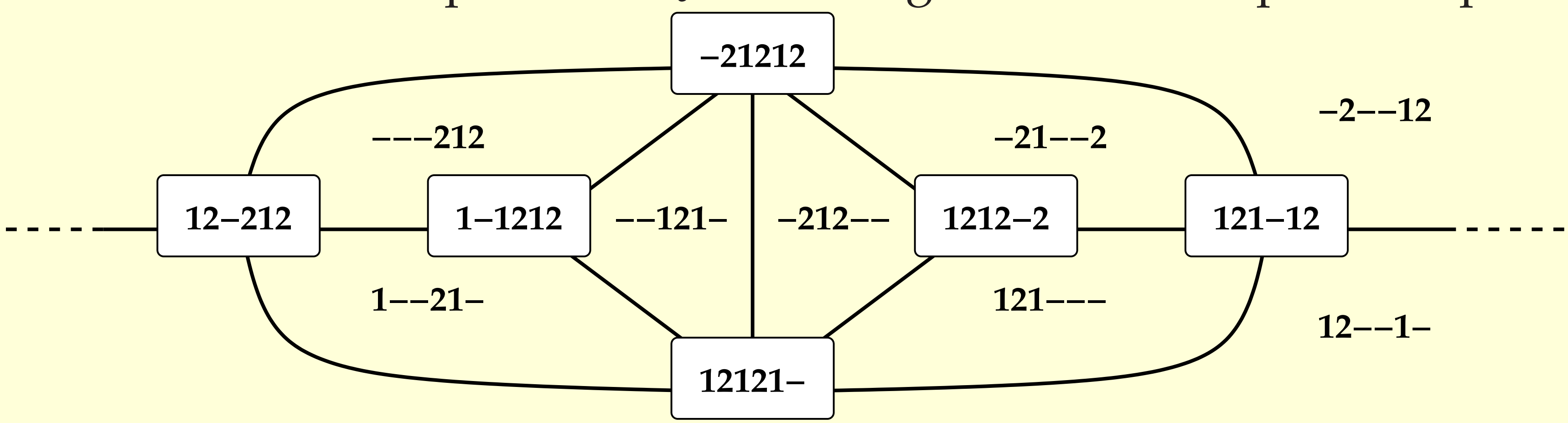
In the slides to come, we will give *canonical* resolutions of Richardson varieties (and thus of projected Richardsons too), not dependent on choices of Q and R.

Example: Brion's "log resolutions" of the Richardson stratification of GL_3/B.

Let $Q = R = 121$, reduced words in S_3, so $Q\overleftarrow{R} = 121121$. Then the dual complex is a 2-sphere:



The vertices are labeled with the complements of letters, the regions with reduced subwords with product w_0. $R = 212$ gives an isomorphic complex:



Moduli spaces of stable maps of rational curves.

Fix a 2-homology class $\beta \in H_2(M)$ and a number n of "marked points". We consider maps $\gamma : \Sigma \to M$, where Σ is a tree of smooth $\mathbb{P}^1$s with simple normal (i.e. nodal) crossings and n points (not at the nodes) marked $1 \ldots n$. Also we require $\gamma_*([\Sigma]) = \beta$. (The 0 in "$\overline{\mathcal{M}}_{0,n}$" below is for the only genus we consider.)

Call the map γ **stable** if Σ has only finitely many automorphisms compatible with γ. Specifically, each component of Σ collapsed by γ to a point should have at least three nodes + marked points.

There is a natural topology on this space $\overline{\mathcal{M}}_{0,n}(M, \beta)$ of maps, making it compact (in limits, Σ can break). It is more naturally a stack than a scheme, in that one should remember the finite automorphism groups.

Theorem [Fulton-Pandharipande '95]. $\overline{\mathcal{M}}_{0,n}(G/P, \beta)$ is a smooth proper stack, or in other language, a compact orbifold.

This space comes with an sncd, consisting of the reducible Σ.

Already the case $\overline{\mathcal{M}}_{0,n}(pt, 0)$ is interesting. Here D has one component for each of the $2^{n-1} - n - 1$ nontrivial divisions of the marked points. The classical cross-ratio gives an isomorphism $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, where the sncd is the values $0, 1, \infty$. In particular the sncd is not anticanonical.

(Now the new stuff!) A moduli space of equivariant maps.

We define a locally closed substack $\overline{\mathcal{M}}'$. Assume Σ's components come in a chain ·OOO···O·, not in a knottier tree. Put a $\mathbb{G}_m$ action on Σ, speed 1 on each component, with opposed weights $+1, -1$ at the two tangent lines at each node. The two $\mathbb{G}_m$-fixed points in Σ at the ends, with respective tangent weights $+1, -1$, we **mark** and call $0, \infty \in \Sigma$ (note in particular that $n \geq 2$).

If a circle acts on M, together we get a T^2-action on $\overline{\mathcal{M}}'_{0,n}(M, \beta)$. The fixed points $\overline{\mathcal{M}}'_{0,n}(M, \beta)^{\mathbb{G}_m}$ for the diagonal are the circle-equivariant stable maps.

Theorem. $\overline{\mathcal{M}}'_{0,n\leq 3}(G/P, \beta)^{\mathbb{G}_m}$ is a smooth stack (albeit disconnected).

Fix a regular dominant weight, say $\check{\rho}$, acting on G/P; by regularity $(G/P)^{\check{\rho}} \cong W/W_P$ with Białynicki-Birula decompositions the Bruhat and opposite Bruhat decompositions.

Let $\beta = \left[\, \overline{\check{\rho} \cdot x} \,\right] \in H_2(G/P)$ where $x \in X_u^v$ is general in the Richardson variety.

Theorem. Let $\widetilde{X}_u^v(m) = \left\{ \gamma \in \overline{\mathcal{M}}'_{0,m+2}(G/P, \beta)^{\mathbb{G}_m} \;:\; \gamma(0) = uP/P,\ \gamma(\infty) = vP/P \right\}$. Then $\widetilde{X}_u^v(m)$ is smooth, connected, and for $m \leq 1$ is proper. The map $\widetilde{X}_u^v(1) \to X_u^v$ taking $\gamma \mapsto \gamma(\text{the marked point} \neq 0, \infty)$ is a resolution of singularities.

Effectively, we're not just specifying a class in homology $H_2(G/P)$, but in equivariant homology $H_2^{\mathbb{G}_m}(G/P)$.

Main theorems: the sncd $D \subset \widetilde{X}_u^v(0)$.

Theorem. 1. Let $\gamma : \Sigma \to X_u^v$ lie in our space $\widetilde{X}_u^v(0)$, and enumerate Σ's fixed points $p_0 = 0, p_1, \ldots, p_c = \infty \in \Sigma^{\mathbb{G}_m}$ so that p_{i-1}, p_i lie in the same component of Σ for $i = 1 \ldots c$. Then $\gamma(p_1) < \ldots < \gamma(p_{c-1})$ in the open Bruhat interval (u, v).

2. The substack of $\widetilde{X}_u^v(0)$ consisting of stable curves through $w_1 < \ldots < w_{c-1}$ in the open Bruhat interval (u, v) is isomorphic to $\prod_{i=1}^{c} \widetilde{X}_{w_{i-1}}^{w_i}(0)$, and in particular is smooth of codimension $c - 1$. (Here we take $w_0 = u$, $w_c = v$.)

3. Hence the substack D consisting of reducible stable curves is sncd, and in the G/B case, is anticanonical.

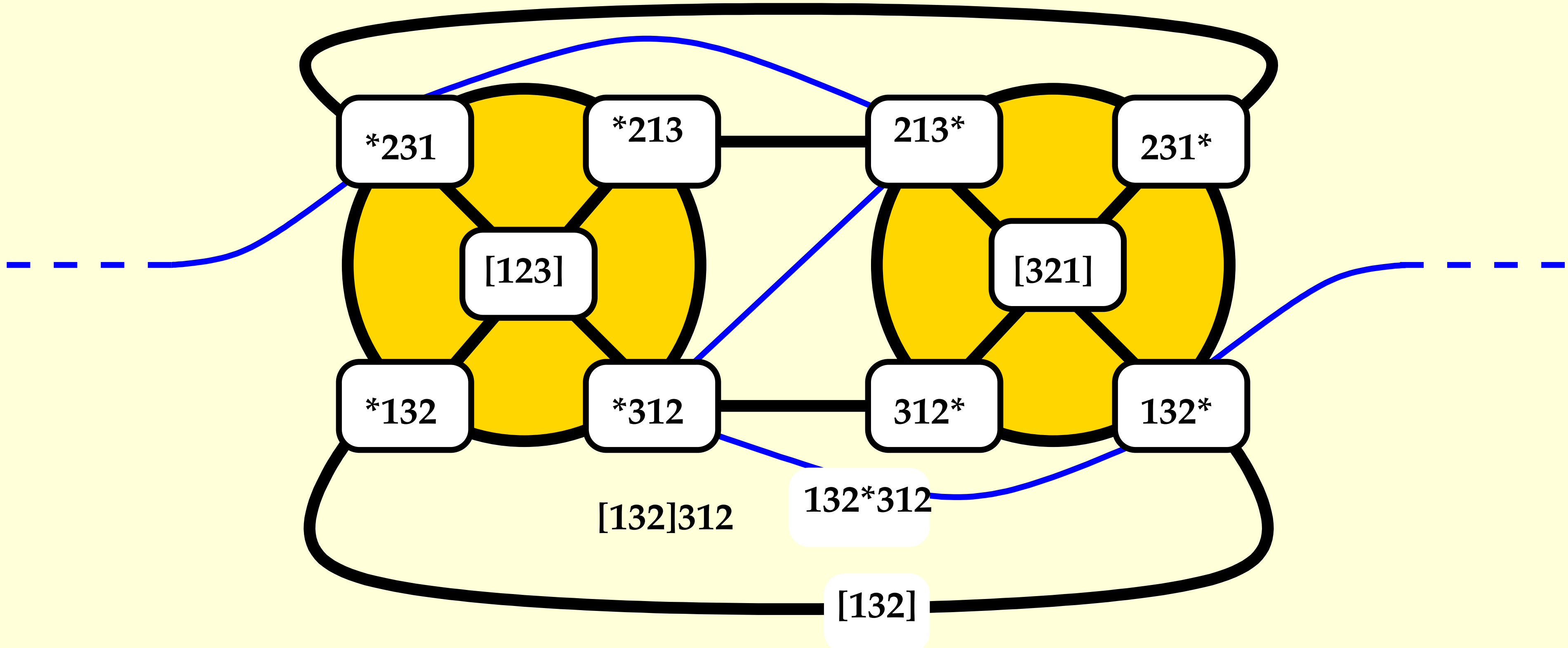
4. $\widetilde{X_u^v}(1) \cong \widetilde{X_u^v \times \mathbb{P}^1}(0)$. (This doesn't quite work for higher n.)

#3 prompts us to consider D's dual complex, which is exactly the order complex of the Bruhat interval (u, v). This simplicial complex was proven in [Björner-Wachs '82] to be homeomorphic to a sphere, using "EL-shellability". Another case confirmed of the folklore conjecture!

By #4, the dual of the sncd for $\widetilde{X}_u^v(1)$ is almost the suspension of the Björner-Wachs sphere – first cross with an interval, triangulate, *then* cone the ends.

Note that one can define $\widetilde{X}_u^v(n)$ using stable maps into X_u^v rather than into G/B; we only used maps into G/B to more easily prove smoothness. The singular variety X_u^v already contains the seeds of its resolution!

Example: the dual complex $\Delta(D)$ to the sncd D in $\widetilde{X}^{321}_{123}(1)$.



In each component of D, Σ breaks into OO, with the marked point on one of the two components. Each corresponding vertex of $\Delta(D)$ is labeled by γ(the node).

When the component with the marked point collapses, taking the node with it, we [box] its image. Otherwise the $*$ specifies the component of the marked point. A few of the bigger faces of $\Delta(D)$ are also labeled.

The link of the $[u]$ (or $[v]$) vertex is a copy of the Björner-Wachs sphere. Deleting those (gold) balls gives a (blue) triangulation of their sphere times an interval.

GKM spaces and the Grassmannian case.

Call a torus action d-**GKM** (for Goresky-Kottwitz-MacPherson) if it fixes only finitely many subvarieties of dimension $\leq d$ (necessarily toric). [GKM '98] only considered $d = 1$, which includes flag manifolds G/P. The fixed points and curves in a 1-GKM space give the vertices and edges of a graph.

It is easy to see that if M is d-GKM, then each $\overline{\mathcal{M}}_{0,n}(M, \beta)$ is $(d-1)$-GKM. For example, the isolated fixed points in $\widetilde{X}^{v}_{u}(0)$ consist of chains of covers of T-fixed curves, each connecting some w_i to $w_{i+1} = w_i r_\delta$.

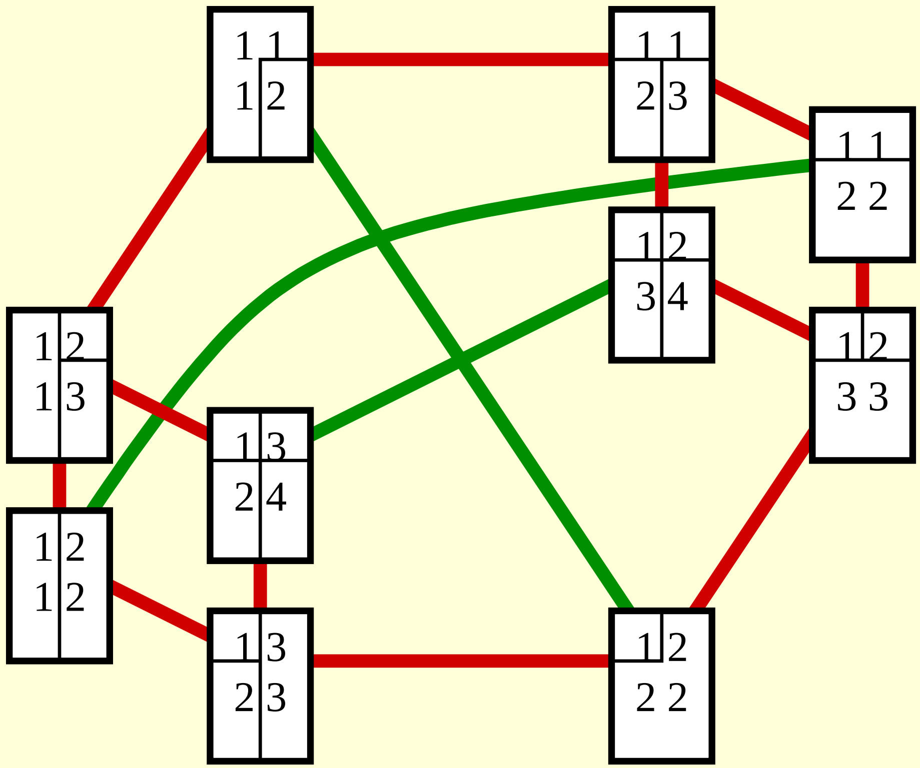
[Guillemin-Zara '01] observed that Grassmannians are 2-GKM, which they called "3-independence" (of isotropy weights). Hence each $\widetilde{X}^{\nu}_{\mu}(0)$ is 1-GKM.

To describe its GKM graph, we need recall the combinatorial notion of **rim-hook tableau** of shape μ/ν. This is a chain $\mu = \lambda_0 \subset \lambda_1 \subset \ldots \subset \lambda_m = \nu$ of partitions, where each λ_i/λ_{i-1} is a **rim-hook**, i.e. connected and containing no 2×2 square.

Theorem. The T-fixed points on $\widetilde{X}^{\nu}_{\mu}(0)$ correspond to rim-hook tableaux $\{\tau\}$. Most of the edges out of τ involve breaking a rim-hook into two or gluing two together, making τ'. If rim-hooks i and $i+1$ of τ together contain a 2×2 square (so can't be glued), or share no boundary (ditto), the resulting union has a canonical alternate breaking, τ'. These pairs (τ, τ') are the graph edges.

Example: the GKM graph for $\widetilde{X}^{2+2}_{\emptyset}(0)$.

To the jth rim-hook we associate a root $\beta_j := e_\ell - e_r$ where $\ell, r \in [n]$ are the diagonals of the ends of the rim-hook (e.g. $r = \ell + 1$ for single squares). Draw the GKM graph nicely by placing τ at position $\Phi(\tau) := \sum_{\boxed{i},\, \boxed{j}} \mathrm{sign}(j - i)\, \beta_j$.



In this example the edges for gluing-*or*-cutting rim-hooks are red, those for gluing-*then*-rebreaking-the-other-way edges are green.

WARNING: in larger examples this function Φ is not injective.

Bonus: computing the isotropy weights on $\widetilde{X}^{\gamma}_{\mu}(0)$, up to scale.

Let T act on the 1-GKM space M, and $\rho : \mathbb{G}_m \to T$ a regular coweight ($M^{\rho} = M^T$).

A T-fixed curve δ in $\overline{\mathcal{M}}_{0,n}(M, \beta)^{\mathbb{G}_m}$ is a family $(\gamma_t)_{t \in \mathbb{P}^1}$ of $\mathbb{G}_m$-equivariant stable maps $\gamma_t : \Sigma_t \to M$, the union of whose images forms a toric T-invariant surface $S \subseteq M$. The images $\gamma_t(0)$ and $\gamma_t(\infty)$ are constant in t, and are the sink and source of the $\mathbb{G}_m$-action on S.

Let λ, μ be the isotropy weights on $T_{\gamma_t(0)} S$. Then the coweight lattice of $Stab_T(\delta)$ is $(\lambda^{\perp} \cap \mu^{\perp}) + \mathbb{Z}\rho$, whose perp is $(\mathbb{Z}\lambda + \mathbb{Z}\mu) \cap \rho^{\perp}$.

The isotropy weights of T on $\gamma_0, \gamma_\infty \in \delta$ lie in $+\mathbb{N}\lambda - \mathbb{N}\mu$ and $-\mathbb{N}\lambda + \mathbb{N}\mu$ respectively, whose intersections with $\rho^{\perp}$ are $\cong \mathbb{N}$. We have thus determined those isotropy weights up to scale.

In the case $M = Gr(k, n)$, the possible S boil down to (here $a < b < c < d$)

- $Gr(1, \mathbb{C}^{abc})$, gluing two rim-hooks along a horizontal edge
- $Gr(2, \mathbb{C}^{abc})$, gluing two rim-hooks along a vertical edge
- $Gr(1, \mathbb{C}^{ab}) \times Gr(1, \mathbb{C}^{cd})$, swapping nonoverlapping rim-hooks
- $Gr(1, \mathbb{C}^{ac}) \times Gr(1, \mathbb{C}^{bd})$ or $Gr(1, \mathbb{C}^{ad}) \times Gr(1, \mathbb{C}^{bc})$, gluing then rebreaking.

I computed each isotropy weight with the recipe above, then invented Φ, which I set up so the isotropy weight would be a multiple of $\Phi(\tau) - \Phi(\tau')$.

Q: $\exists$ an equivariant ample line bundle $\mathcal{L}$ on $\widetilde{X}^{\gamma}_{\mu}(0)$ with $\Phi(\tau) = T\text{-}wt(\mathcal{L}|_{\tau})$?