Semiorthogonal Decompositions and Dimension

- X, Y smooth projective varieties $/k$ $(=\mathbb{C})$

$\mathrm{Coh}(X) \subset D^b_{\mathrm{coh}}(X) = D^b(\mathrm{Coh}(X))$ k-linear Δ-cat

- All functors exact, k-linear.

<u>Theorem</u>. If $\exists$ fully faithful

$$F: D^b_{\mathrm{coh}}(X) \longrightarrow D^b_{\mathrm{coh}}(Y),$$

then $\dim X \leq \dim Y$.

<u>Orlov's Conjecture</u>. $\underbrace{\mathrm{RDim}}_{\text{Rouquier dimn}}(D^b_{\mathrm{coh}}(X)) = \dim X$

Notation: Generation time for Δ-Cats

T Δ-category

$S \subset \operatorname{ob} T$

Going to build objects of T from S using

Operations:

\$0 $\longleftarrow$ (1) Shifts, $\oplus$, extracting direct summands

\$1 $\longleftarrow$ (2) Given: $A \to B \to C \to A[1]$

it costs \$1 to build B out of A, C.

$\langle S \rangle_{N+1}$ = objects built from S costing $\leq$ \$$N$.

$$\operatorname{RDim}(T) = \inf \{ d \in \mathbb{R} \mid \exists\, G \in \operatorname{ob} T \text{ s.t. } \langle G \rangle_{d+1} = T \}.$$

- $\operatorname{RDim}(T) = \infty$ very possible

$$\dim(X) \le \operatorname{RDim}(D^b_{coh}(X)) \le 2\dim(X)$$

Rouquier

Bondal Van den Bergh

Orlov conjectures $=$.

R ~~[illegible]~~ m, reg local ring of dim d

$R/m \notin \langle R \rangle_{d}$

Examples. 1. R Dedekind domain (i.e. $k[t]$)

$D^b(\text{f.g. } R\text{-modules}) = \langle R \rangle_2 \implies \operatorname{RDim}(\ ) \leq 1.$

$K \in$ $\qquad \operatorname{Ext}^{\geq 2}_R(-,-) = 0$

$\implies \quad K = \bigoplus_{i \in \mathbb{Z}} H^i(K)[-i] \rightsquigarrow$ STS fg $R\text{-Mod} \subset \langle R \rangle_2$

M fg R-Mod

$$0 \to \underset{\text{direct summand of } R^{\oplus N}}{\underline{P}} \to R^{\oplus n} \to M \to 0$$

$\$0 \quad \$0 \quad \$1$

$\implies$ done.

2. X sm. proj curve. $\mathcal{L}/X$ of degree $\geq 8g_X$

$\underline{\text{Theorem}}$ (Orlov) $D^b_{Coh}(X) = \langle \underline{\mathcal{L}^{-1} \oplus \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2}} \rangle_2$

STS $Coh(X) \subset \langle \{\mathcal{L}^{-1}, \mathcal{O}, \mathcal{L}, \mathcal{L}^{\otimes 2}\} \rangle_2$

$\underline{\text{I}}$ngredient: $\underline{\text{I}}$f $\mathcal{E}$ vector bundle $\mu_{min}(\mathcal{E}) \geq 4g$

then $\exists$ exact sequence

$$\mathcal{L}^{-1\,\oplus a} \longrightarrow \mathcal{O}^{\oplus b} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\{\underline{(\mathcal{L}^{-1})^{\oplus b} \longrightarrow \mathcal{O}^{\oplus a}} \longrightarrow T \longrightarrow 0$$

Lemma. $F: T \to T'$ fully faithful w/ left adjoint L, then $\mathrm{RDim}\, T \leq \mathrm{RDim}\, T'$.

So Orlov's Conjecture $\Longrightarrow$ Theorem.

Examples. $f: Y \to X$ $\quad Rf_* \mathcal{O}_Y = \mathcal{O}_X$

then $Lf^*: D^b_{coh}(X) \to D^b_{coh}(Y)$ is fully faithful

$1 \Rightarrow Rf_* Lf^*$

(1) $\mathcal{O}_X \xrightarrow{=} f_* \mathcal{O}_Y$

(2) $R^i f_* \mathcal{O}_Y = 0 \quad i > 0$.

– ∃ examples due to Kuznetsov of fully faithful functors $D^b_{coh}(K3) \longrightarrow D^b_{coh}$(cubic fourfold)

$$2 \leq 4$$

– Any faithful functor $D^b_{coh}(X) \longrightarrow D^b_{coh}(Y)$ is of the form

$$K \xmapsto{\Phi_E} Rpr_{2*}\left(Lpr_1^* K \overset{L}{\underset{O_{X\times Y}}{\otimes}} E \right)$$

for some $E \in D^b_{coh}(X \times Y)$

Φ_E fully faithful $\Longleftrightarrow$ $Ext^*_{O_Y}(E_{x_1}, E_{x_2}) = Ext^*_{O_X}(k(x_1), k(x_2))$ $\forall x_1, x_2 \in X$

Results.

Theorem! $D^b_{coh}(X) = \langle \{\mathcal{O}_X(n)\}_{n<0} \rangle_{\dim X + 1}$

Defn (Pirozhkov). T Δ-Cat,

$$CRdim(T) = \inf \{ d \in \mathbb{Z} \mid \exists S \subset \mathrm{ob}\, T \text{ countable s.t. } \langle S \rangle_{d+1} = T \}$$

$\shortparallel$

countable Rouquier dimn

k uncountable

Theorem 1 + ϵ $\Longrightarrow$ $CRDim(D^b_{coh}(X)) = \dim X$

$\Longrightarrow$ If $\exists$ $F : D^b_{coh}(X) \longrightarrow D^b_{coh}(Y)$ fully faithful, then $\dim X \leq \dim Y$.

Theorem 2. $K_0 \to K_1 \to \cdots \to K_{\dim X+1}$ in $D^b_{coh}(X)$ s.t. $\forall i$ $K_i \to K_{i+1}$ is zero on cohomology sheaves. $\Longrightarrow$ $K_0 \xrightarrow{0} K_{\dim X+1}$ is zero.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d, "\sim"] & \mathcal{O}_X[d] \\ \mathcal{F}_1[1] \arrow[r] & \mathcal{F}_2[2] \arrow[r] & \cdots \arrow[ul, "\sim"] \end{tikzcd}$$

$\Longleftrightarrow \xi \in H^d(X, \mathcal{O}_X)$

R regular ring of dim d

$K_0 = K \in D^b(\text{f.g. } R\text{-mod})$

$$K_1 \longrightarrow \oplus R^{d_i}[-i] \longrightarrow K_0 \xrightarrow{+}$$

surjection on coho sheaves

$K_0, K_1, \ldots, K_d \in \langle R \rangle_0$