Varieties of general type with doubly exponential asymptotics

(joint work with Burt Totaro and Chengxi Wang)

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Varieties of General Type

Note: Throughout, we'll work over $\mathbb{C}$.

A smooth algebraic variety X is **of general type** if for any sufficiently large integer ℓ, the rational map

$$\phi_{\ell K_X} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(\ell K_X)))$$

is birational onto its image.

Question

For X of general type, how large does ℓ have to be before $\phi_{\ell K_X}$ is birational onto its image?

Boundedness of Pluricanonical Maps

There is a uniform bound on ℓ in each dimension:

Theorem (Hacon–McKernan, Takayama, Tsuji, 2006)

For every positive integer n, there exists an integer r_n such that if X is a smooth complex projective variety of general type and dimension n, then $\phi_{\ell K_X} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(\ell K_X)))$ is birational onto its image for $\ell \geq r_n$.

New Question

How large are the bounds r_n?

Bounds in Low Dimension

- $r_1 = 3$
  - For C a smooth curve of genus $g \geq 2$, $\mathcal{O}_C(3K_C)$ is very ample
- $r_2 = 5$ (Bombieri, 1973)
  - The extreme example is $X_{10} \subset \mathbb{P}^3(5, 2, 1, 1)$
- $27 \leq r_3 \leq 57$ (lower bound: Iano-Fletcher, 2000; upper bound: J. Chen, M. Chen, 2008)
  - Most extreme known example is the desingularization of $X_{46} \subset \mathbb{P}^4(23, 7, 6, 5, 4)$
- For larger n, no upper bounds known!

Volume

The **volume** of a smooth variety X of general type and dimension n is

$$\mathrm{vol}(X) = \limsup_{\ell \to \infty} \frac{h^0(X, \mathcal{O}_X(\ell K_X))}{(\ell^n/n!)} > 0.$$

When K_X is nef, $\mathrm{vol}(X) = K_X^n$.

Corollary (to HMTT Theorem)

For every positive integer n, there exists a positive constant a_n such that if X is a smooth complex projective variety of general type and dimension n, then $\mathrm{vol}(X) \geq a_n$.

In fact, we can prove:

$$a_n \geq \frac{1}{(r_n)^n}.$$

Volume Bounds in Low Dimension

- $a_1 = 2$
  - The extreme example is $\mathrm{vol}(C) = 2$ for a smooth curve C of genus 2
- $a_2 = 1$
  - An extreme example is $X_{10} \subset \mathbb{P}(5, 2, 1, 1)$, with volume 1
- $\frac{1}{1680} \leq a_3 \leq \frac{1}{420}$
  - The desingularization of $X_{46} \subset \mathbb{P}(23, 7, 6, 5, 4)$ has the smallest known volume, $\frac{1}{420}$

Asymptotic Bounds on Varieties of General Type

Question

How do the constants r_n, a_n behave for n large?

Theorem (Ballico, Pignatelli, Tasin, 2013)

- *For $n \geq 7$, $r_n \geq \frac{n(n-3)}{9}$.*
- *For $n \geq 5$, $a_n < \frac{3^{n+1}}{n^n}$.*

Theorem (ETW, 2021)

For every positive integer $n \geq 3$:

- $r_n > 2^{2^{(n-2)/2}}$
- $a_n < \frac{1}{2^{2^{n/2}}}$

Example: $r_{10} > 3 \times 10^6$

A Conjecture on Pairs

Conjecture (Kollár)

The klt pair of general type with standard coefficients and minimum volume is

$$(X,\Delta) = \left(\mathbb{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{s_{n+1}-1}{s_{n+1}}H_{n+1}\right),$$

with $\mathrm{vol}(K_X+\Delta) = \frac{1}{(s_{n+2}-1)^n} < \frac{1}{2^{2^n}}$.

- (X,Δ) is **of general type** if $K_X+\Delta$ is a big $\mathbb{Q}$-Cartier divisor
- $H_0,\ldots H_{n+1}$ are general hyperplanes in $\mathbb{P}^n$
- (X,Δ) has **standard coefficients** if each coefficient of Δ is of the form $1-\frac{1}{j}$ for $j \in \mathbb{Z}_+$
- $s_0, s_1, \ldots$ is **Sylvester's sequence**, defined $s_0 = 2$, $s_m = s_0 \cdots s_{m-1} + 1$ for each $m \geq 1$

A Conjecture on Pairs, cont.

- Pairs (X, Δ) arise as **global quotients** of varieties of general type Y by a group of birational automorphisms
  - $K_Y = f^*(K_X + \Delta)$
  - Δ encodes the ramification data in codimension 1
- ETW Theorem + Conjecture would show that a_n is roughly between $\frac{1}{2^{2^{n/2}}}$ and $\frac{1}{2^{2^n}}$
- Conjecture only known for $n = 1$, where the extremal example $(\mathbb{P}^1, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2)$ with $\mathrm{vol}(K_X + \Delta) = \frac{1}{42}$ is the **Hurwitz orbifold**

Weighted Projective Space

The **weighted projective space** $Y = \mathbb{P}(a_0, \ldots, a_N)$ is the quotient variety $(\mathbb{A}^{N+1} \setminus 0)/\mathbb{G}_m$, where $\mathbb{G}_m$ acts as

$$t \cdot (x_0, \ldots, x_N) = (t^{a_0} x_0, \ldots, t^{a_N} x_N).$$

Y is **well-formed** if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_N) = 1$.

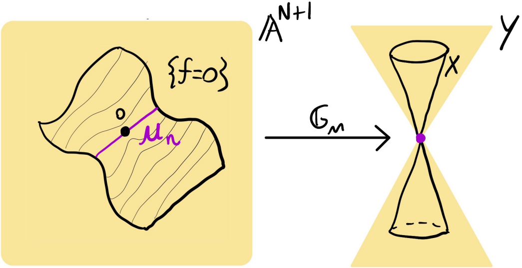
- Y is toric, hence rational
- Y has only cyclic quotient singularities
- For each $d \in \mathbb{Z}$, there is a reflexive sheaf $\mathcal{O}_Y(d)$ associated to a Weil divisor on Y

Examples:

- $\mathbb{P}(1, 1, \ldots, 1) = \mathbb{P}^N$
- $\mathbb{P}(1, 1, 2)$ is the cone over a conic curve in $\mathbb{P}^2$

Quasi-smooth Hypersurfaces

A hypersurface $X = \{f = 0\}$ of (weighted) degree d in $\mathbb{P}(a_0, \ldots, a_N)$ is **quasi-smooth** if its affine cone $\{f = 0\} \subset \mathbb{A}^{N+1}$ is smooth away from $0 \in \mathbb{A}^{N+1}$.



If X is quasi-smooth,

- X has only cyclic quotient singularities
- The adjunction formula $K_X = \mathcal{O}_X(d - a_0 - \cdots - a_N)$ holds

A Criterion for Quasi-smoothness

Proposition (Iano-Fletcher, 2000)

A general hypersurface X of degree d in the weighted projective space $\mathbb{P}(a_0, \ldots, a_N)$ is quasi-smooth if and only if one of the following properties holds:

1. $a_i = d$ *for some* i, *or*
2. *for each nonempty* $I \subset \{0, \ldots, N\}$, *either*
   - (a) d *is an* $\mathbb{N}$*-linear combination of weights* a_i *for* $i \in I$, *or*
   - (b) *there are at least* $|I|$ *numbers* $j \notin I$ *such that* $d - a_j$ *is an* $\mathbb{N}$*-linear combination of the numbers* a_i *with* $i \in I$.

- Easy case: If all weights $a_0, \ldots, a_N$ divide d, the general hypersurface of degree d is quasi-smooth
- Idea of proof: $I \leftrightarrow T$-invariant stratum of $\mathbb{P}(a_0, \ldots, a_N)$ with nonzero coordinates in I

Canonical Models

Let X be a variety with canonical class $\mathbb{Q}$-Cartier and $f : W \to X$ a desingularization with

$$K_W = f^* K_X + \sum_j b_j E_j.$$

X has **canonical singularities** (**terminal singularities**) if all $b_j \geq 0$ $(b_j > 0)$.

$$X \dashrightarrow X_{\min} \dashrightarrow X_{\mathrm{can}}$$

$\uparrow$	$\uparrow$	$\uparrow$
smooth, general type, K_X big	terminal singularities, K_X nef	canonical singularities, K_X ample

All three models have the same plurigenera $h^0(X, \mathcal{O}_X(\ell K_X))$, volume, image of $\phi_{\ell K_X}$, etc.!

The Reid-Tai Criterion

A **cyclic quotient singularity of type** $\frac{1}{r}(b_1, \ldots, b_s)$ is the quotient $\mathbb{A}^s/\mu_r$, where $\zeta \in \mu_r$ acts as $\zeta(t_1, \ldots, t_s) = (\zeta^{b_1} t_1, \ldots, \zeta^{b_s} t_s)$.

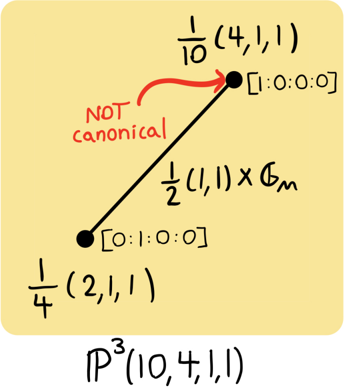
Theorem (Reid-Tai Criterion)

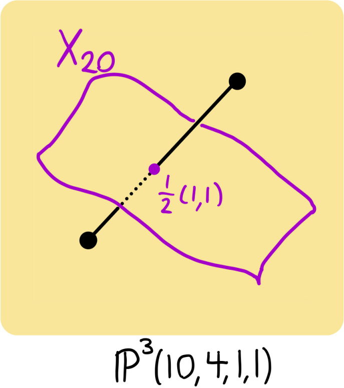
Assume that the quotient singularity $\frac{1}{r}(b_1, \ldots, b_s)$ *is* **well-formed** *in the sense that* $\gcd(r, b_1, \ldots, \widehat{b_j}, \ldots, b_s) = 1$ *for all* $j = 1, \ldots, s$. *Then the quotient singularity is canonical (terminal) if and only if*

$$\sum_{j=1}^{s} ib_j \bmod r \geq r$$

$(> r)$ *for all* $i = 1, \ldots, r-1$.

Example: $X_{20} \subset \mathbb{P}^3(10, 4, 1, 1)$





Summary

General hypersurface X of degree d in $\mathbb{P}(a_0, \ldots, a_N)$

$\rightsquigarrow$ Calculations on $d, a_0, \ldots, a_N$

- Is X quasi-smooth? If yes:
- Does X have canonical/terminal singularities?
- $K_X = \mathcal{O}_X(d - a_0 - \cdots - a_N)$ (we will want $K_X = \mathcal{O}_X(1)$)
- If $K_X = \mathcal{O}_X(1)$,
  - $\mathrm{vol}(X) = \frac{d}{a_0 \cdots a_N}$
  - $\phi_{\ell K_X}$ is not birational onto its image for $\ell < \max\{a_0, \ldots, a_n\}$

Performing the Optimization

- Spring 2021: Choose weights as products of all subsets of a set of consecutive integers $\implies a_n < 1/e^{n(\log n)^2}$ (Totaro, Wang)
- Summer 2021: Choose weights as products of $k-1$ of the first k primes (or their powers) $\implies a_n < 1/e^{0.99n^{3/2}(\log n)^{1/2}}$ (Esser, Tao, Totaro, Wang)
  - Use an equidistribution result on unit circle to optimize constant in the exponent
- Fall 2021: ??? $\implies a_n < 1/2^{2^{n/2}}$

Moving Beyond General Type

Idea: Consider **Calabi-Yau** X rather than general type ($K_X \sim_{\mathbb{Q}} 0$).

Lemma

Suppose $X_d \subset \mathbb{P}(a_0, \ldots, a_N)$ *is quasi-smooth with* $d = a_0 + \cdots + a_N$. *Then* X *has canonical singularities.*

Proof: By the adjunction formula, $K_X = \mathcal{O}_X(d - a_0 \cdots - a_N) = \mathcal{O}_X$. For $f : W \to X$ a desingularization, we have

$$K_W = f^* K_X + \sum_j b_j E_j = \sum_j b_j E_j.$$

Cyclic quotient singularities are klt $\implies b_j > -1$ $\implies b_j \geq 0 \implies X$ has canonical singularities.

Constructing Examples with Sylvester's Sequence

Sylvester's Sequence: $2, 3, 7, 43, 1807, \ldots$

- Terms are pairwise relatively prime
- Grows doubly exponentially with m: $s_m > 2^{2^{m-1}}$
- $\frac{1}{s_0} + \cdots + \frac{1}{s_m} = 1 - \frac{1}{s_0 \cdots s_m} = \frac{1}{s_{m+1}-1}$
- This series $\sum_m \frac{1}{s_m}$ converges to 1 more quickly than for any other sequence of reciprocals (Soundararajan, 2005)

Key Example

For every dimension n, the general weighted projective hypersurface of degree $d = s_0 \cdots s_n = s_{n+1} - 1$ in

$$\mathbb{P}^{n+1}(d/s_0, d/s_1, \ldots, d/s_n, 1)$$

is well-formed, quasi-smooth, and has canonical singularities with $K_X = \mathcal{O}_X$.

Canonical Calabi-Yau Varieties

Theorem (ETW)

In every dimension $n \geq 2$, there is a canonical Calabi-Yau variety X and an ample Weil divisor A on X such that $\mathrm{vol}(A) < \frac{1}{2^{2^n}}$.

Conjecture (ETW)

For a positive integer n, let $d = (2s_n - 3)(s_n - 1)$. A general hypersurface X of degree d in

$$\mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-1}, s_n - 1, s_n - 2)$$

is the canonical Calabi-Yau n-fold with ample Weil divisor $\mathcal{O}_X(1)$ of minimal volume.

Remark: A positive lower bound exists for each n (Birkar, 2020).

Low-dimensional Calabi-Yau Examples

- $n = 1$: $X_6 \subset \mathbb{P}^2(3, 2, 1)$ with $\mathrm{vol}(\mathcal{O}_X(1)) = 1$
  - Elliptic curve E embedded in Proj of the section ring $R(E, \mathcal{O}(P))$, where P is the origin of E
- $n = 2$: $X_{66} \subset \mathbb{P}^3(33, 22, 6, 5)$ with $\mathrm{vol}(\mathcal{O}_X(1)) = 1/330$
  - Lowest volume among all canonical K3 surfaces with an ample Weil divisor (Brown, 2007)
- $n = 3$: $X_{3486} \subset \mathbb{P}^4(1743, 1162, 498, 42, 41)$ with $\mathrm{vol}(\mathcal{O}_X(1)) = 1/498240036$
- $n = 4$:

  $$X_{6521466} \subset \mathbb{P}^5(3260733, 2173822, 931638, 151662, 1806, 1805)$$

  with $\mathrm{vol}(\mathcal{O}_X(1)) \approx 2.0 \times 10^{-24}$
  - Last two examples have lowest volume of $\mathcal{O}_X(1)$ among all Calabi-Yau weighted projective hypersurfaces of dimension $3, 4$ (Brown, Kasprzyk, 2016)

Terminal Fano Varieties

Theorem (ETW)

For every integer $n \geq 3$, there is a terminal Fano variety X of dimension n with $\mathrm{vol}(-K_X) < \frac{1}{2^{2^n}}$.

Conjecture (ETW)

For each integer $n \geq 2$, let $d = (2s_{n-1} - 3)(s_{n-1} - 1)$. Then a general hypersurface X of degree d in

$$\mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-2}, s_{n-1} - 1, s_{n-1} - 2, 1)$$

is the terminal Fano n-fold of minimal anticanonical volume.

Remark: A positive lower bound exists for each n (Birkar, 2019).

Low-dimensional Fano Examples

- $n = 2$: $X_6 \subset \mathbb{P}^3(3,2,1,1)$ with $\mathrm{vol}(-K_X) = 1$
  - Smooth del Pezzo surface of degree 1 embedded with its anticanonical ring
- $n = 3$: $X_{66} \subset \mathbb{P}^4(33,22,6,5,1)$ with $\mathrm{vol}(-K_X) = 1/330$
  - Lowest anticanonical volume among all terminal Fano 3-folds (J. Chen, M. Chen, 2008)
- $n = 4$: $X_{3486} \subset \mathbb{P}^4(1743,1162,498,42,41,1)$ with $\mathrm{vol}(-K_X) = 1/498240036$
  - Lowest anticanonical volume among all terminal Fano weighted projective hypersurfaces of dimension 4 with $K_X = \mathcal{O}_X(-1)$ (Brown, Kasprzyk, 2016)

Doubling Weights

Beginning with our key Calabi-Yau example

$$X_{s_0 \cdots s_m} \subset \mathbb{P}(d/s_0, d/s_1, \ldots, d/s_m, 1) \text{ (canonical CY)},$$

double degree and repeat weights twice:

$$X_{2s_0 \cdots s_m} \subset \mathbb{P}((d/s_0)^{(2)}, (d/s_1)^{(2)}, \ldots, (d/s_m)^{(2)}, 1^{(2)}) \text{ (terminal CY)}.$$

Then remove a 1:

$$X_{2s_0 \cdots s_m} \subset \mathbb{P}((d/s_0)^{(2)}, (d/s_1)^{(2)}, \ldots, (d/s_m)^{(2)}, 1) \text{(canonical gen. type)}$$

This gives a general type example in odd dimensions $n = 2m + 1$; other similar examples exist for even n.

Examples of General Type

Theorem (ETW)

Let n be a positive integer $n \geq 4$. If $n = 2m+1$ is odd, let $d = s_{m+1} - 1$, and X be a general hypersurface X of degree $2d$ in

$$\mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_m)^{(2)}, 1).$$

If n is even, let $d = (s_m - 1)(2s_m - 1)$, and X be a general hypersurface of degree $2d$

$$\mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_{m-1})^{(2)}, 2(s_m - 1), (s_m - 1)^{(2)}, 1).$$

Then a desingularization W of X is smooth of general type with $\mathrm{vol}(W) < \frac{1}{2^{2^{n/2}}}$ and $\phi_{\ell K_W}$ is not birational for $\ell \leq 2^{2^{(n-2)/2}}$.

This proves the main theorem for smooth varieties of general type.

Thank you!