Toroidalization principles for klt singularities

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Stanford AG Seminar

Outline

1. Regional fundamental group

2. Toroidalization of the fundamental group

3. Connections with termination of flips

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The regularity of a n-dimensional klt singularity is an integer in the interval $\{0, \dots, n-1\}$.

Regularity

Let (X, Δ) be a log pair. The *model regularity* of (X, Δ) is the maximum number of components $S_1, \ldots, S_r \subset \lfloor \Delta \rfloor$ which are $\mathbb{Q}$-Cartier so that $S_1 \cap \cdots \cap S_r \neq \emptyset$.

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The regularity of a n-dimensional klt singularity is an integer in the interval $\{0, \ldots, n-1\}$. Given a n-dimesional klt singularity of regularity r, we may say it is *r-regular*.

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If $(T, \Delta_T; t)$ is a n-dimensional toric singularity, then $\operatorname{reg}(T, \Delta_T; t) = n - 1$.

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Theorem (M, 2021)

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3. *there exists* $B_Y \geqslant 0$ *supported on* $S_1 \cup \dots \cup S_{r+1}$ *for which*

$$\pi_*\colon \pi_1^{\mathrm{reg}}(Y, B_Y; y) \to \pi_1^{\mathrm{reg}}(X, B + M; x)$$

has cokernel of order at most $c(n)$*.*

Jordan property vs regularity

Corollary (M, 2021)

Let $(X, \Delta; x)$ be a n-dimensional r-regular klt singularity. Then, there exists a short exact sequence

$$1 \to A \to \pi_1^{\mathrm{reg}}(X, \Delta; x) \to N \to 1,$$

where A is abelian of rank at most $r+1$ and N has order at most $c(n)$.

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The previous conjecture is known as the upper bound for minimal log discrepancies.

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Conjecture (Upper bound for the minimal log discrepancy)

Let $(X, \Delta; x)$ be a klt singularity. Then, there exists a prime divisor E over X with center x for which $a_E(X, \Delta; x) \leqslant a(n)$.

The previous conjecture is known as the upper bound for minimal log discrepancies.

Conjecture (Zariski closedness of the diminished base locus)

Let (X, Δ) be a projective generalized klt pair. Then, $\mathrm{Bs}_{-}(K_X + \Delta)$ is Zariski closed.

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Fundamental groups and termination

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Theorem (M, 2021, work in progress)

Assume the following conjectures hold:

1. *The boundedness of the regional fundamental group in dimension n,*
2. *the upper bound for the minimal log discrepancy in dimension n, and*
3. *the Zariski closedness of the diminished base locus in dimension n.*

Then, termination of flips with scaling for generalized pairs in dimension n holds.

Thanks for your attention!