# Toroidalization principles for klt singularities

J. Moraga

Princeton University.

Stanford AG Seminar

#### Outline

Regional fundamental group

Toroidalization of the fundamental group

3 Connections with termination of flips

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The regularity of a n-dimensional klt singularity is an integer in the interval  $\{0, \ldots, n-1\}$ .



Let  $(X,\Delta)$  be a log pair. The *model regularity* of  $(X,\Delta)$  is the maximum number of components  $S_1,\ldots,S_r\subset [\Delta]$  which are  $\mathbb{Q}$ -Cartier so that  $S_1\cap\cdots\cap S_r\neq\varnothing$ .

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The singularity  $(X, \Delta)$  is exceptional if and only if  $\operatorname{reg}(X, \Delta; x) = 0$ . If  $(T, \Delta_T; t)$  is a n-dimensional toric singularity, then  $\operatorname{reg}(T, \Delta_T; t) = n - 1$ .



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- **1** there exists  $B_Y \geqslant 0$  supported on  $S_1 \cup \cdots \cup S_{r+1}$  for which

$$\pi_* : \pi_1^{\text{reg}}(Y, B_Y; y) \to \pi_1^{\text{reg}}(X, B + M; x)$$

has cokernel of order at most c(n).



#### Jordan property vs regularity

#### Corollary (M, 2021)

Let  $(X,\Delta;x)$  be a n-dimensional r-regular klt singularity. Then, there exists a short exact sequence

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where A is abelian of rank at most r+1 and N has order at most c(n).

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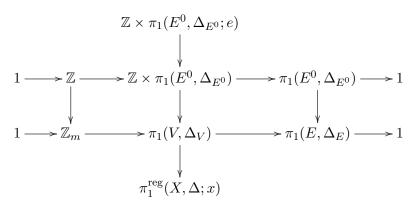
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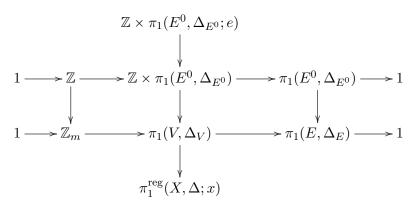
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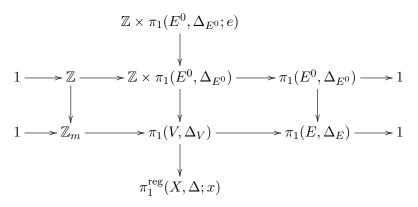


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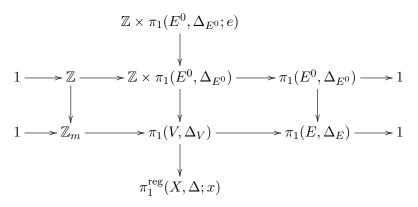
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If  $m \ge c(n, N)$ , then the birational regularity of (X, B) is at least r.

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#### Definition

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#### Theorem (M, 2021, work in progress)

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## Conjecture (Zariski closedness of the diminished base locus)

Let  $(X, \Delta)$  be a projective generalized klt pair. Then,  $\mathrm{Bs}_-(K_X + \Delta)$  is Zariski closed.

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## Fundamental groups and termination

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#### Theorem (M, 2021, work in progress)

Assume the following conjectures hold:

- lacktriangledown The boundedness of the regional fundamental group in dimension n,
- $oldsymbol{2}$  the upper bound for the minimal log discrepancy in dimension n, and
- $oldsymbol{0}$  the Zariski closedness of the diminished base locus in dimension n.

Then, termination of flips with scaling for generalized pairs in dimension n holds.

Thanks for your attention!