Logarithmic resolution of singularities via multi-weighted blow-ups

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Joint work-in-progress with Dan Abramovich
(some results in an earlier work [Que20] by the speaker (arXiv: 2005.05939))

Main Theorem A

Theorem (Logarithmic embedded resolution in characteristic zero)

Let X be a closed, reduced substack of a smooth, (strict) toroidal Artin stack Y over a field $\Bbbk$ of characteristic zero.
Then there exists a sequence of multi-weighted blow-ups

$$\Pi\colon Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = Y$$

with successive proper transforms $X_i \subset Y_i$ of X such that:

(i) *X_N is a smooth, (strict) toroidal Artin stack over $\Bbbk$, and so are each Y_i.*

(ii) *Π is an isomorphism over the logarithmically smooth locus $X^{\text{log-sm}}$ of X.*

(iii) *$\Pi^{-1}(X \smallsetminus X^{\text{sm}})$ is a snc (= simple normal crossings) divisor on X_N.*

This procedure $(X \subset Y) \mapsto (X_N \subset Y_N)$ is functorial with respect to logarithmically smooth morphisms of such pairs.

Main Theorem B

Theorem (Our singularity invariant drops after each multi-weighted blow-up)

Let $X \subset Y$ be as before, but X is logarithmically singular.
Then there exists a multi-weighted blow-up $\pi \colon Y' \to Y$ with proper transform $X' \subset Y'$ of X such that:

(i) *Y' is a smooth, (strict) toroidal Artin stack over $\Bbbk$.*

(ii) $\max \mathrm{inv}(X' \subset Y') < \max \mathrm{inv}(X \subset Y)$.

(iii) *π is an isomorphism away from the closed substack of X consisting of points $p \in |X|$ such that $\mathrm{inv}_p(X \subset Y) = \max \mathrm{inv}(X \subset Y)$.*

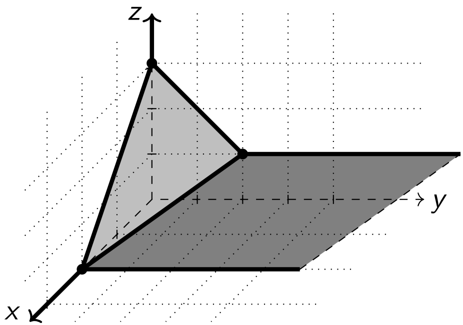
This procedure is functorial with respect to logarithmically smooth, surjective morphisms of pairs.

Multi-weighted blow-ups

(via an example, where one multi-weighted blow-up resolves all singularities)

Example

Take $I = (x^2 + \underline{y}^2\underline{z} + \underline{z}^3) \subset \Bbbk[x, \underline{y}, \underline{z}]$, and set $X = V(I) \subset \mathbb{A}^3_{x,\underline{y},\underline{z}} = Y$.

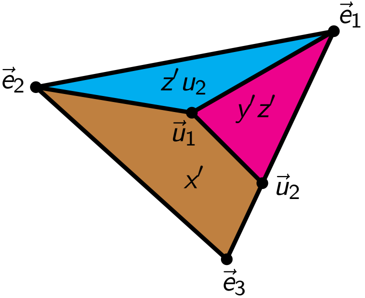


We will see later: The algorithm instructs us to take the multi-weighted blow-up of $\mathbb{A}^3$ along $J = (x^2, \underline{y}^2\underline{z}, \underline{z}^3)$.

The first step is to sketch the Newton polytope P of J.

Multi-weighted blow-ups

(via an example, where one multi-weighted blow-up resolves all singularities)



From the Newton polytope P of J, we get a cross-section of the normal fan Σ of J, where $\vec{u}_1 = (3, 2, 2)$ and $\vec{u}_2 = (1, 0, 2)$ are the normal vectors to the shaded facets of P.

This leads to a homomorphism $\beta \colon \mathbb{Z}^5 \to \mathbb{Z}^3$, which maps

$$\vec{e}_i \mapsto \begin{cases} \vec{e}_i, & \text{if } 1 \leq i \leq 3 \\ \vec{u}_1, & \text{if } i = 4 \\ \vec{u}_2, & \text{if } i = 5 \end{cases}$$

Multi-weighted blow-ups

(via an example, where one multi-weighted blow-up resolves all singularities)

β fits into the following exact sequence:

$$0 \to \mathbb{Z}^2 \xrightarrow{\alpha = \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 2 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}} \mathbb{Z}^5 \xrightarrow{\beta = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}} \mathbb{Z}^3 \to 0$$

Lift the normal fan Σ on $\mathbb{Z}^3$ to a smooth fan $\widehat{\Sigma}$ on $\mathbb{Z}^5$ generated by $\{\widehat{\sigma} \colon \sigma \in \Sigma\}$, where $\widehat{\sigma} :=$ cone spanned by those $\vec{e}_i \in \mathbb{Z}^5$ with $\beta(\vec{e}_i) \in \sigma$.

Definition (Multi-weighted blow-up of $\mathbb{A}^3$ along J)

$$\mathscr{B}\mathrm{l}_J(\mathbb{A}^3) := [X_{\widehat{\Sigma}} \,/\, G_\beta] \xrightarrow{\pi_J} \mathbb{A}^3$$

where: $X_{\widehat{\Sigma}}$ is the smooth toric variety associated to $\widehat{\Sigma}$,

$G_\beta = \ker(\mathbb{G}_m^5 = T_{\mathbb{Z}^5} \xrightarrow{T_\beta} T_{\mathbb{Z}^3} = \mathbb{G}_m^3) \simeq \mathbb{G}_m^2$, and
π_J is induced by the toric morphism $X_{\widehat{\Sigma}} \to \mathbb{A}^3$ defined by β.

Multi-weighted blow-ups

(via an example, where one multi-weighted blow-up resolves all singularities)

Explicitly,

$$\mathscr{B}l_J(\mathbb{A}^3) = \left[\left(\mathrm{Spec}(\Bbbk[x', \underline{y}', \underline{z}', \underline{u}_1, \underline{u}_2]) \smallsetminus V(x', \underline{y}'\underline{z}', \underline{z}'\underline{u}_2)\right) \ / \ \mathbb{G}_m^2\right] \xrightarrow{\pi_J} \mathbb{A}^3_{x,\underline{y},\underline{z}}$$

where:

(i) The irrelevant ideal $(x', \underline{y}'\underline{z}', \underline{z}'\underline{u}_2)$ can be determined from the maximal cones of Σ.

(ii) From the matrix $\beta = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$, we see that π_J is induced by $x \mapsto x'\underline{u}_1^3\underline{u}_2$, $\underline{y} \mapsto \underline{y}'\underline{u}_1^2$ and $\underline{z} \mapsto \underline{z}'\underline{u}_1^2\underline{u}_2^2$.

(iii) The rows of $\alpha = \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 2 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$ are the $\mathbb{Z}^2$-weights of $x', \underline{y}', \underline{z}', \underline{u}_1, \underline{u}_2$.

(iv) $\underline{u}_1 = 0$ and $\underline{u}_2 = 0$ define the exceptional divisors on $\mathscr{B}l_J(\mathbb{A}^3)$.

(v) Finally, we have indicated the logarithmic structure on $\mathscr{B}l_J(\mathbb{A}^3)$ — it is "dictated by that on $\mathbb{A}^3$ and the exceptional divisors", under which the Artin stack $\mathscr{B}l_J(\mathbb{A}^3)$ is smooth + logarithmically smooth over $\Bbbk$.

Multi-weighted blow-ups

(via an example, where one multi-weighted blow-up resolves all singularities)

The total transform of $I = (x^2 + \underline{y}^2\underline{z} + \underline{z}^3)$ is:

$$\begin{aligned} I \cdot \mathscr{O}_{\mathscr{B}l_J(\mathbb{A}^3)} &= ((x'\underline{u}_1^3\underline{u}_2)^2 + (\underline{y}'\underline{u}_1^2)^2(\underline{z}'\underline{u}_1^2\underline{u}_2^2) + (\underline{z}'\underline{u}_1^2\underline{u}_2^2)^3) \\ &= \underline{u}_1^6\underline{u}_2^2 \cdot \underbrace{\left(x'^2 + \underline{y}'^2\underline{z}' + \underline{z}'^3\underline{u}_2^4\right)}_{:= \text{ weak transform } I' \text{ of } I}. \end{aligned}$$

We claim $V(I') \subset \mathscr{B}l_J(\mathbb{A}^3)$ is smooth + logarithmically smooth over $\Bbbk$. Indeed, compute the "logarithmic derivatives" of $x'^2 + \underline{y}'^2\underline{z}' + \underline{z}'^3\underline{u}_2^4$:

$$\mathscr{D}^{\leq 1}_{Y',\log}(I') = (x', \underline{y}'^2\underline{z}', \underline{z}'^3\underline{u}_2^4) = (1)$$

where the last equality follows since all three generators are units on the x'-chart, $\underline{y}'\underline{z}'$-chart, or $\underline{z}'\underline{u}_2$-chart.

Multi-weighted blow-ups

Further remarks

(i) Multi-weighted blow-ups are examples of "*fantastacks*" in the sense of Geraschenko-Satriano.

(ii) Multi-weighted blow-ups admit *good moduli spaces*: in the example above, the good moduli space of $\mathscr{B}\mathrm{l}_J(\mathbb{A}^3)$ is the toric variety X_Σ.

(iii) Generally, multi-weighted blow-ups can be done along a "monomial ideal", or more generally, a "monomial Rees algebra" (by passing to a Veronese subalgebra which is generated in degree 1).

(iv) Earlier, we have only indicated how to define the *weak transform* under a multi-weighted blow-up.
Analogous to the classical case, the *proper transform* is obtained by "factoring out as many exceptional divisors from each individual element of the total transform of I".

Goal: Define $\mathrm{inv}_p(X \subset Y)$ in Main Theorem B. More generally, we will define $\mathrm{inv}_p(I)$ for any ideal $I \subset \mathscr{O}_Y$ and $p \in |V(I)| =: |X|$.

Smooth locally, every smooth, toroidal Artin stack over $\Bbbk$ looks like a smooth, strict toroidal $\Bbbk$-scheme. Therefore, we may assume Y is a smooth, strict toroidal $\Bbbk$-scheme. Let D be its snc divisor.

The singularity invariant inv

Setting things up

At any closed point $p \in Y$, we may choose a regular system of parameters

$$\underbrace{x_1, \ldots, x_{n-r}}_{\text{ordinary parameters}} \ , \ \underbrace{\underline{x}_{n-r+1}, \ldots, \underline{x}_n}_{\text{monomial parameters}}$$

such that $D = (\underline{x}_{n-r+1} \cdots \underline{x}_n = 0)$ at p.

We have the sheaf $\mathscr{D}^{\infty}_{Y,\log}$ of logarithmic differential operators on Y, locally generated over $\mathscr{O}_Y$ by

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-r}}, x_{n-r+1}\frac{\partial}{\partial x_{n-r+1}}, \ldots, x_n\frac{\partial}{\partial x_n}.$$

These are precisely the differential operators preserving the ideal of D. We denote by $\mathscr{D}^{\leq m}_{Y,\log}$ the logarithmic differential operators of order $\leq m$.

Definition (Logarithmic order)

$$\text{log-ord}_p(I) = \min\{n \in \mathbb{N} \colon \mathscr{D}^{\leq n}_{Y,\log}(I)_p = \mathscr{O}_{Y,p}\} \in \mathbb{N} \sqcup \{\infty\}$$

where $\min(\emptyset) := \infty$.

Note that this is a stronger invariant than the usual order:

Example

Consider $X = V(I) \subset \mathbb{A}^2_{x,\underline{y}}$, where $I = (\underline{y} - x^2) \subset \Bbbk[x, \underline{y}]$. X is evidently smooth (i.e. $\text{ord}_p(I) = 1$ everywhere), but $\text{log-ord}_{(0,0)}(I) = 2$, i.e. X is not logarithmically smooth over $\Bbbk$. Problem is: X is not transverse to the logarithmic stratum $V(\underline{y})$!

Definition (Maximal contact elements)

Suppose $1 \leq a := \text{log-ord}_p(I) < \infty$. Then a (logarithmic) *maximal contact element* of I at p is an element of $\mathscr{D}^{\leq a-1}_{Y,\log}(I)$ which has logarithmic order 1, i.e. is part of a system of ordinary parameters at p.

Note that since $\text{char}(\Bbbk) = 0$, $\mathscr{D}^{\leq 1}_{Y,\log}(\mathscr{D}^{\leq a-1}_{Y,\log}(I))_p = \mathscr{D}^{\leq a}_{Y,\log}(I)_p = (1)$, i.e. we can always find a local anti-derivative of 1 — so maximal contact elements always exist locally at p.

Definition (Monomial saturation)

$$\mathscr{M}(I) := \mathscr{D}^{\infty}_{Y,\log}(I)$$

This is a *monomial ideal* *with respect to the logarithmic structure on Y*, in the sense that it is locally generated by monomials in monomial parameters. It is also the *smallest* monomial ideal containing I.

Example

(a) If $I = (\underline{y} - x^2) \subset \Bbbk[x, \underline{y}]$, then $\mathscr{M}(I) = (1)$.

(bi) If $I = (\underline{x}^2 - \underline{y}^2 z) \subset \Bbbk[\underline{x}, \underline{y}, z]$, then $\mathscr{M}(I) = (\underline{x}^2, \underline{y}^2)$.

(bii) If $I = (\underline{x}^2 - \underline{y}^2 \underline{z}) \subset \Bbbk[\underline{x}, \underline{y}, \underline{z}]$, then $\mathscr{M}(I) = (\underline{x}^2, \underline{y}^2 \underline{z})$.

We now use the three notions above to define inv. To do this, we first associate to the pair $(I, p \in X = V(I))$ the following auxiliary data:

(a) a sequence of natural numbers $(b_1, \ldots, b_k)$,

(b) a sequence of ordinary parameters $(x_1, \ldots, x_k)$, and

(c) a monomial ideal $Q \subset \mathscr{O}_{Y,p}$.

(a) and (b) are defined inductively, with (c) defined only at the end of the induction.

Base case

(i) If $\text{log-ord}_p(I) = \infty \Leftrightarrow p \in V(\mathscr{M}(I))$, set $k = 0$, and $Q := \mathscr{M}(I)_p$.

(ii) If not, set $b_1 := \text{log-ord}_p(I) \in \mathbb{N}_{>0}$ and $x_1 :=$ a maximal contact element of I at p. We also set $I[1] := I$, and proceed to the inductive step.

Inductive step

Now assume that for some $1 \leq \ell < k$, we have defined the sequences $(b_1, \ldots, b_\ell)$ and $(x_1, \ldots, x_\ell)$, and we have an ideal $I[\ell] \subset \mathscr{O}_{V(x_1,\ldots,x_{\ell-1})}$.

Naïve approach: Consider $I[\ell]|_{x_\ell=0}$ and define $b_{\ell+1}$, $x_{\ell+1}$ (and possibly Q) based on $\text{log-ord}_p(I[\ell]|_{x_\ell=0})$.

Problem: The sequence $(b_1, \ldots, b_k)$ and Q at the end depends on the choice of $(x_1, \ldots, x_k)$.

Idea (Hironaka, Włodarczyk): "$\mathscr{D}$-saturate" (or "$\mathscr{D}$-balance" in the words of Kollár) the ideal $I[\ell]$ so that étale locally it "looks the same" along any choice of maximal contact hypersurface $x_\ell = 0$.

Definition (Coefficient ideals)

If $1 \leq b := \max \text{log-ord}(I) < \infty$, the (logarithmic) coefficient ideal of I is

$$\mathscr{C}(I, b) := \left(\prod_{j=0}^{b-1} \mathscr{D}^{\leq j}_{Y,\log}(I)^{c_j} \colon c_j \in \mathbb{N},\ \sum_{j=0}^{b-1} c_j(b-j) \geq b! \right)$$

i.e. the $(b!)^{\text{th}}$ graded piece of the $\mathscr{O}_Y$-subalgebra of $\mathscr{O}_Y[t]$ generated by

$$I \cdot t^b \qquad \mathscr{D}^{\leq 1}_{Y,\log}(I) \cdot t^{b-1} \qquad \mathscr{D}^{\leq 2}_{Y,\log}(I) \cdot t^{b-2} \qquad \ldots \qquad \mathscr{D}^{\leq b-1}_{Y,\log}(I) \cdot t^1.$$

In particular, note $\mathscr{D}^{\leq b-1}_{Y,\log}(I)^{b!} \subset \mathscr{C}(I, b)$, so given any maximal contact element x of I, $x^{b!} \in \mathscr{C}(I, b)$.

Inductive step (continued)

Set

$$I[\ell+1] := \mathscr{C}(I[\ell], b_\ell)|_{x_\ell=0} \subset \mathscr{O}_{V(x_1,\dots,x_\ell)}.$$

(i) If $\text{log-ord}_p(I) = \infty$, set $k = \ell$ and $Q := \mathscr{M}(I[\ell+1])_p$.

(ii) If not, set $b_{\ell+1} := \text{log-ord}_p(I[\ell+1])$, and $x_{\ell+1} :=$ a maximal contact element of $I[\ell+1]$ at p.
Repeat the inductive step, with input $(b_1, \dots, b_\ell, b_{\ell+1})$, $(x_1, \dots, x_\ell, x_{\ell+1})$, and $I[\ell+1]$.

Lemma (Q)

Both $(b_1, \dots, b_k)$ and Q do not depend on the choice of $(x_1, \dots, x_k)$.

Definition (The singularity invariant inv)

$$
\mathrm{inv}_p(I) := \begin{cases} \left(b_1, \frac{b_2}{(b_1-1)!}, \frac{b_3}{\prod_{j=1}^{2}(b_j-1)!}, \dots, \frac{b_k}{\prod_{j=1}^{k-1}(b_j-1)!}\right) & \text{if } \mathscr{Q} = (0); \\ \left(b_1, \frac{b_2}{(b_1-1)!}, \frac{b_3}{\prod_{j=1}^{2}(b_j-1)!}, \dots, \frac{b_k}{\prod_{j=1}^{k-1}(b_j-1)!}, \infty\right) & \text{if } \mathscr{Q} \neq (0). \end{cases}
$$

We usually denote its finite entries by $a_1, \dots, a_k$.

We well-order the set of all invariants of ideals at points by the lexicographic order, but with a caveat: it considers a truncation of a sequence to be strictly larger than the sequence itself.
E.g., $(1,1,1) < (1,1) < (1,2,5) < (1,3,3) < (1,\infty) < (1)$.

Definition (Maximum invariant)

$$\max \operatorname{inv}(I) := \max_{p \in X = V(I)} \operatorname{inv}_p(I)$$

Lemma (Q)

1. $\operatorname{inv}_p(I) = (1, 1, \ldots, 1)$ *of length equal to height of* $I_p \iff V(I)$ *is smooth, toroidal at* p*.*
2. $\operatorname{inv}_p(I)$ *is upper semi-continuous on* Y*.*
3. *If* $f\colon \widetilde{Y} \to Y$ *is logarithmically smooth, and* $f\colon \widetilde{p} \mapsto p$*, then* $\operatorname{inv}_{\widetilde{p}}(I \cdot \mathscr{O}_{\widetilde{Y}}) = \operatorname{inv}_p(I)$*. If* f *is moreover surjective, then* $\max \operatorname{inv}(I) = \max \operatorname{inv}(I \cdot \mathscr{O}_{\widetilde{Y}})$*.*

Blow-up center

Local definition

Definition (Local description of blow-up center)

Fix $p \in X$ such that $\mathrm{inv}_p(I) = \max \mathrm{inv}(I)$. Set

$$J(I,p) := \left(x_1^{a_1}, \ldots, x_k^{a_k}, Q^{1/d}\right) \subset \mathscr{O}_{Y,p}[t]$$

where $d := \prod_{j=1}^{k} (b_j - 1)!$.

This is an integrally closed Rees algebra defined locally around p, given by the integral closure in $\mathscr{O}_{Y,p}[t]$ of the $\mathscr{O}_{Y,p}$-subalgebra generated by $x^{a_i N_i} \cdot t^{N_i}$ and $Q \cdot t^d$. Here, each $N_i \in \mathbb{N}_{>0}$ is chosen sufficiently large so that $a_i N_i \in \mathbb{N}_{>0}$.

Note that the d^{th} Veronese subalgebra of $J(I,p)$ is generated in degree 1.

Blow-up center

Global definition

Lemma (Q)

There exists a Rees algebra $J(I)$ on Y such that for every $p \in Y$,

$$J(I)_p := \begin{cases} J(I,p), & \text{if } \operatorname{inv}_p(I) = \max \operatorname{inv}(I) \\ \mathscr{O}_{Y,p}[t], & \text{if } \operatorname{inv}_p(I) < \max \operatorname{inv}(I) \end{cases}$$

To be clear, $J(I)$ is our *blow-up center* on Y with respect to I.

Lemma (Q)

(i) *$J(I) \supset$ Rees algebra of I, i.e. $J(I)$ is I-admissible. (In fact, a "unique admissibility property" holds for $J(I)$.)*

(ii) *If $f\colon \widetilde{Y} \to Y$ is logarithmically smooth and surjective, then $J(I \cdot \mathscr{O}_{\widetilde{Y}}) = J(I) \cdot \mathscr{O}_{\widetilde{Y}}$.*

Multi-weighted blow-up along center

Theorem (Abramovich-Q)

There is a "multi-weighted blow-up of Y along $J(I)$"

$$\pi_{J(I)} \colon \mathscr{B}\mathrm{l}_{J(I)} Y \to Y$$

such that:

(i) *π is an isomorphism away from the closed locus of points $p \in Y$ such that $\mathrm{inv}_p(I) = \max \mathrm{inv}(I)$.*

(ii) *locally above points $p \in Y$ such that $\mathrm{inv}_p(I) = \max \mathrm{inv}(I)$, π restricts to the multi-weighted blow-up of Y along $J(I, p)$.*

Invariant drops

Theorem (Abramovich-Q)

Let I' be the weak transform of I under $\pi_{J(I)}$. Then

$$\max \operatorname{inv}(I') < \max \operatorname{inv}(I).$$

Since the proper transform $\widetilde{I}$ of I contains I', we get $\max \operatorname{inv}(\widetilde{I}) \leq \max \operatorname{inv}(I') < \max \operatorname{inv}(I)$ — which is Main Theorem B.