Pre-talk: Say a map of stacks $[X/G] \to [Y/H]$ is an equivariant map if it arises from the data ($G \to H$, $X \overset{G}{\circlearrowright} \to Y$)

Can every map $[X/G] \to [Y/H]$ be factored as a span of equivariant maps

$$\begin{array}{ccc} & [P/K] & \\ \swarrow\sim & & \searrow \\ [X/G] & & [Y/H] \end{array} \quad ?$$

Line bundles in equivariant elliptic cohomology $/\mathbb{C}$

(joint w/ D. Berwick-Evans)

Given G a compact Lie group & $G \curvearrowright M$ a (finite-type) G-manifold,

Defn. [Grojnowski, (BE-T)]

$\mathrm{Ell}_G M$ is a sheaf on $\mathrm{Bun}_G E$.

E an elliptic curve (or family of elliptic curves, e.g. $\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathcal{M}_{1,1} \end{array}$) $/\mathbb{C}$

2 ways to view $\mathrm{Bun}_G E$: [Narasimhan-Seshadri, Atiyah-Bott, Donaldson]

$\boxed{DG} \xleftrightarrow{\text{GIT}} AG$

[S-equivalence]

{moduli of semistable $G_{\mathbb{C}}$-bundles on E }

$\pi_0(\text{moduli}) \simeq \pi_1 G$

$\bigcup$ $\bigcup$

{ flat G-bundles$/_E$ } $\longleftrightarrow$ { components of the above } $(\pi_1 G)^{\text{tors}}$

{ pairs of commuting elts in G }$^{Ad}/G$

$g = (g_1, g_2)$

$\bigcup$ $\bigcup$ $\bigcup$

$\longleftrightarrow$ { "neutral" component of the above } $\{1\}$

+ " g_1, g_2 may be simultaneously conjugated into a maximal torus of G "

For this talk, $\mathrm{Bun}_G E$ means this last scheme, i.e., just the connected component of the trivial bundle.

We'll first define $(\mathrm{Ell}_G M)$ as a sheaf on $(\mathrm{Bun}_G E)^{an}$.

Defn. $(\mathrm{Ell}_G M)^{an}_g := H^{\bullet,[an]}_{G^g}(M^g; \mathbb{C}[\beta,\beta^{-1}])$

$g = (g_1, g_2)$

G^g = simultaneous centralizer of g_1, g_2.

as a module over $H^{\bullet,an}_{G^g}(pt, \mathbb{C}[\beta,\beta^{-1}]) \simeq \left(\mathcal{O}^{an}_{\mathrm{Bun}_G E}\right)_g$

formal variable

Defn. $H_G(M)$ is the cohomology of ... of deg ~ 2 ...

[Cartan] $\Omega^{dR}_G(M) := \left(C^\infty(\mathfrak{g}_\mathbb{C}; \Omega^\bullet(M))^G, Q \right)$

Rmk polynomials vs. hol functions.

Spec $\mathrm{Poly}(\mathfrak{g}_\mathbb{C})^G$

(Rigidity) $= \mathfrak{g}_\mathbb{C}/G = \mathfrak{t}_\mathbb{C}/W$

DG AG

two elements of $U(1)$ / conj. $\longleftrightarrow$ $\mathrm{Bun}_{\mathbb{G}_m} E = E^\vee \simeq E$

$\mathbb{R} \oplus \mathbb{R} = \mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}$

Ex. $U(1) \curvearrowright \cdot$ $\quad \mathcal{O}_{\mathrm{Bun}_{\mathbb{G}_m} E}$

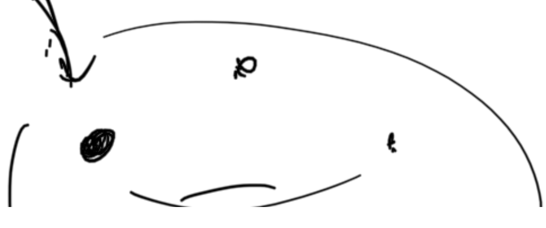
$G \curvearrowright \cdot$ $\quad \mathcal{O}_{\mathrm{Bun}_G E}$ $\quad H_{U(1)}(pt)$

$U(1) \curvearrowright S^2$

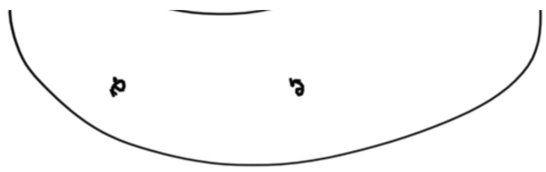
$\to \mathrm{Ell}_{U(1)}(S^2)$ $\quad$ Poly $\mathbb{C}$

$\mathrm{Bun}_{U(1)}(E) \simeq E^\vee$

trivial bundle $\quad g_1, g_2 = 1$ $\qquad M^g = S^2$



$M^g = \{\text{north \& south poles}\}$



$$\mathrm{Ell}_{U(1)}(S^2) = \text{rank 2 v.b.} / \mathrm{Bun}_{U(1)} E$$

$\circ/U(1) \hookrightarrow \mathbb{P}^1/U(1)$ $\quad U(1) \curvearrowright \mathbb{C}$

$\cdot/U(1)$

$\mathbb{C}^\times$

$\mathrm{Ell}_{U(1)} \mathbb{C}$

$\underline{U(1) \curvearrowright U(1)}$

Try to do a Mayer-Vietoris computation.

$$\ker\left(\mathcal{O}_{\mathrm{Bun}_{U(1)} E} \oplus \mathcal{O}_{\mathrm{Bun}_{U(1)} E} \longrightarrow \mathcal{O}_0 \right)$$

$$\mathrm{Ell}_{U(1)} S^2 \simeq \mathcal{O} \oplus \underline{\mathcal{O}(-0)}$$

"$\mathrm{Ell}^{cs}_{U(1)} \mathbb{C}$"

Prop/Defn. $\mathrm{Ell}_G M$ is a (quasi)coherent sheaf of (cdg) algebras on $\underline{\mathrm{Bun}_G E}$. So consider $\underline{\mathrm{Spec}\, \mathrm{Ell}_G M}$.

$\rightsquigarrow$ Many conjectures ("axioms") about this structure [Ginzburg-Kapranov-Vasserot 95, Ando 02], e.g.

Prop. If $[M/G] \xrightarrow{\sim} [N/H]$ is an equivalence of (topological or C^∞) stacks, then $|\mathrm{SpecEll}_G M| \xrightarrow{\sim} \mathrm{SpecEll}_H N$ is a natural iso.

($\therefore$ well-defined functor $\mathrm{TopStk} \xrightarrow{\mathrm{SpecEll}} \mathrm{Sch}/_{\mathcal{M}_{1,1}}$).

Pf. . . .

Line bundles:

Suppose V is an (oriented) rank $2n$ vector bundle on $[M/G]$

(i.e., a G-equivariant vector bundle on M ; or, a map of stacks $[M/G] \to [pt/SO(2n)]$).

Defn. $\mathrm{Thom}(V) \in \mathrm{Pic}\ \mathrm{SpecEll}_G M$.

defined via $\mathrm{Ell}_G^{cvs}(V)$ as a sheaf of modules /

$\mathrm{Ell}_G(M)$ as a sheaf of algebras / $\mathrm{Bun}_G E$.

(Prop. This quasicoherent sheaf is in fact a line bundle.)

Conj. [Ando 02]
Thm. [Ando-Greenlees 11] [BE-T 21]

If V_1 & V_2 are two bundles on $[M/G]$ (with $w_1(V_i) \in H^2_G(M;\mathbb{Z}/2)$ vanishing)

$\longrightarrow$ such that $\underline{p_1(V_1) = p_1(V_2) \in H^4_G(M;\mathbb{Z})}$,

then $\underline{\mathrm{Thom}(V_1)} \simeq \underline{\mathrm{Thom}(V_2)} \in \underline{\mathrm{Pic\,Spec\,Ell}_G M}$.

$(\otimes \omega^{n_1 - n_2})$

Ex. $\underline{U(1) \curvearrowright \mathbb{C}}$ the "standard" representation $\Longrightarrow$ $V = $ std over $*/U(1)$

$\Longrightarrow$ Thom(std) some line $/ \mathrm{Bun}_{U(1)} E \simeq E^{\vee} = E$.

Answer. $\underline{\underline{\mathrm{Thom(std)}}} \simeq \underline{\underline{\mathcal{O}(-\text{zero})}} \simeq \underline{\mathcal{L}^{-1} \quad (\otimes \omega)}$..

Defn. Consider $\sigma(\tau, z) \in \mathcal{O}(\mathbb{H} \times \mathbb{C})$ defined as

$$q = e^{2\pi i \tau} \qquad y = e^{2\pi i z} \qquad \sigma(\tau, z) = \left(e^{\pi i z} - e^{-\pi i z}\right) \prod_{n=1}^{\infty}$$

Note
$$\begin{array}{c} \mathbb{H} \times \mathbb{C} \\ \downarrow \\ \mathrm{Bun}_{U(1)} E \end{array}$$
is an $(SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2)$-Galois cover.

Facts.
$$\sigma(\tau, z + m + n\tau) = (-1)^{m+n} e^{-\pi i (n^2 \tau + 2nz)} \sigma(\tau, z)$$
$$\sigma\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \underline{e^{\pi i cz^2/(c\tau + d)}} \sigma(\tau, z).$$

$\overline{c\tau+d}$

Let these properties define a line bundle $\mathcal{L}$ $(\otimes \omega^{-1})$.

As $\sigma(\tau, z)$ vanishes exactly at the zero section of $\mathcal{E}$,

we discover $\mathcal{O}(\text{zero}) \simeq \mathcal{L}$ $(\otimes \omega^{-1})$.

Ex. $/_{U(1) \times U(1) \times U(1)}$. Now, three "universal" complex line bundles that we'll denote $[std_i]$, $1 \le i \le 3$.

$$\longrightarrow \quad V_1 = [std_1] \otimes [std_2] \otimes [std_3] \oplus \bigoplus_{i=1}^{3} [std_i]$$

$$\longrightarrow \quad V_2 = \left(\bigoplus_{\{i,j\} \subset \{1,2,3\}} [std_i] \otimes [std_j] \right)$$

What's the statement of Ando's conj. now?

Claim. It's that, on $E \times E \times E$, the divisors

$$[p_1 + p_2 + p_3 = 0] + [p_1 = 0] + [p_2 = 0] + [p_3 = 0] \equiv$$

$$[p_1 + p_2 = 0] + [p_1 + p_3 = 0] + [p_2 + p_3 = 0]$$ are rationally equiv.

Pf. 1) [Thm of the cube]

2)

Other questions:

1) a higher-genus categorification?

2) a "direct" proof of Anel's conj?

(i.e.,

$$\begin{tikzcd} KSpin \arrow[r, "?"] \arrow[dr] & H^4(-;\mathbb{Z}) \arrow[d, dashed] \\ & \mathrm{Pic}\,\mathrm{Spec}\,\mathrm{Ell} \end{tikzcd}$$

?)

3) what's going on integrally?