The Grothendieck ring of stacks

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The Grothendieck ring of varieties

Let k be a field. In a 1966 letter to Serre, Grothendieck gave the following definition.

Definition (Grothendieck)

Let $K_0(\mathrm{Var}_k)$ be the abelian group generated by isomorphism classes $\{X\}$ of k-schemes of finite type X, modulo the scissor relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed embedding $Y \hookrightarrow X$.

We make $K_0(\mathrm{Var}_k)$ into a commutative ring with identity:

$$\{X\} \cdot \{Y\} := \{X \times_k Y\}, \qquad 1 = \{\operatorname{Spec} k\}.$$

We denote $\mathbb{L} := \{\mathbb{A}^1\}$ (the Lefschetz class).

Example

If $\pi : E \to X$ is a vector bundle of rank $r \geq 0$, then π is Zariski-locally a product with $\mathbb{A}^r$, hence $\{E\} = \mathbb{L}^r\{X\}$.

Algebraic stacks

Let G be a linear algebraic k-group acting on a k-scheme of finite type X. The action is not necessarily free.

Then there exists a quotient stack $[X/G]$, and the projection $X \to [X/G]$ is a G-torsor (i.e. a principal G-bundle).

If $X = \operatorname{Spec} k$, we get the *classifying stack* $BG := [\operatorname{Spec} k/G]$.

Open/closed substacks of $[X/G]$ all have the form $[Y/G]$, where $Y \subseteq X$ is open/closed and G-invariant.

The Grothendieck ring of stacks

Definition (Ekedahl, 2009)

We define $K_0(\mathrm{Stacks}_k)$ as the abelian group generated by isomorphism classes $\{\mathcal{X}\}$ of algebraic k-stacks of finite type $\mathcal{X}$ (with affine stabilizers), modulo the scissor relations $\{\mathcal{X}\} = \{\mathcal{Y}\} + \{\mathcal{X} \setminus \mathcal{Y}\}$ for every closed embedding $\mathcal{Y} \hookrightarrow \mathcal{X}$, and the vector bundle relations $\{\mathcal{E}\} = \{\mathbb{A}^r_k \times_k \mathcal{X}\}$ for every vector bundle $\mathcal{E} \to \mathcal{X}$ of rank $r \geq 0$.

We make $K_0(\mathrm{Stacks}_k)$ into a commutative ring with identity:

$$\{\mathcal{X}\} \cdot \{\mathcal{Y}\} := \{\mathcal{X} \times_k \mathcal{Y}\}, \qquad 1 = \{\operatorname{Spec} k\}.$$

Example

Vector bundles on BG are of the form $[V/G]$, where V is a G-representation. We have $\{[V/G]\} = \mathbb{L}^{\dim V}\{BG\}$.

Computations

Here are some examples of computations of $\{BG\}$.

Example

Let $G = \mathbb{G}_{\mathrm{m}}$. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{A}^1$ by scalar multiplication. Then

$$\mathbb{L}\{B\mathbb{G}_{\mathrm{m}}\} = \{[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]\} = \{[(\mathbb{A}^1 \setminus 0)/\mathbb{G}_{\mathrm{m}}]\} + \{B\mathbb{G}_{\mathrm{m}}\} = 1 + \{B\mathbb{G}_{\mathrm{m}}\}$$

$$\Rightarrow \{B\mathbb{G}_{\mathrm{m}}\}(\mathbb{L}-1) = 1 \Rightarrow \{B\mathbb{G}_{\mathrm{m}}\} = (\mathbb{L}-1)^{-1} = \{\mathbb{G}_{\mathrm{m}}\}^{-1}.$$

Example

Let $G = \mu_n = \operatorname{Spec} k[x]/(x^n - 1)$ act on $\mathbb{A}^1$ by scalar multiplication. Then

$$\mathbb{L}\{B\mu_n\} = \{[\mathbb{A}^1/\mu_n]\} = \{B\mu_n\} + \{[(\mathbb{A}^1 \setminus 0)/\mu_n]\} = \{B\mu_n\} + \mathbb{L} - 1$$

$$\Rightarrow (\mathbb{L}-1)\{B\mu_n\} = (\mathbb{L}-1) \Rightarrow \{B\mu_n\} = 1.$$

Relation with $K_0(\mathrm{Var}_k)$

Theorem (Ekedahl)

The canonical ring homomorphism $K_0(\mathrm{Var}_k) \to K_0(\mathrm{Stacks}_k)$ induces an isomorphism

$$K_0(\mathrm{Stacks}_k) \cong K_0(\mathrm{Var}_k)[\frac{1}{\mathbb{L}}, \frac{1}{\mathbb{L}^n - 1} : n \geq 1].$$

Example

$$\{B\,\mathrm{GL}_n\} = \{\mathrm{GL}_n\}^{-1} = \prod_{i=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^i)^{-1}.$$

$$\{[X/\,\mathrm{GL}_n]\} = \{X\}\{\mathrm{GL}_n\}^{-1}.$$

Relation to stable rationality of BG

Definition

We say that BG is stably rational if for some generically free G-representation V, the rational quotient V/G is stably rational (i.e. $V/G \times_k \mathbb{A}^r_k \sim_{\mathrm{bir}} \mathbb{A}^s_k$ for some $r, s \geq 0$).

This definition does not depend on the choice of V. It is closely related to Noether's problem (rationality of fields of invariants).

Many examples and results indicate a strict relation between the stable rationality of BG and the equality $\{BG\} = 1$ (G finite) or $\{BG\}\{G\} = 1$ (G connected).
For finite G, both conditions imply vanishing of the unramified Brauer group of $k(V)^G$, V any faithful representation of G.

Relation to stable rationality of BG

Question

1) (Ekedahl) If G is finite, is it true that $\{BG\} = 1$ if and only if BG is stably rational?
2) (Talpo - Vistoli) If G is connected, is it true that $\{BG\}\{G\} = 1$ if and only if BG is stably rational?

1) true e.g. for μ_n, S_n, subgroups of $\mathrm{GL}_3(\mathbb{C})$ (when $k = \mathbb{C}$), many counterexamples to Noether's problem.
2) true e.g. for
$G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{PGL}_3, G_2, \mathrm{Spin}_7, \mathrm{Spin}_8$, ...

Counterexamples

Assume k perfect, $\operatorname{char} k \neq 2$, and that there exists a biquadratic field extension K/k. Let E_1, E_2, E_3 be the distinct quadratic subextensions of K/k.

Let $T := R^{(1)}_{E_1 \times E_2/k}(\mathbb{G}_{\mathrm{m}})$ (a k-torus of rank 3), and let $A := T[2]$ be the 2-torsion subgroup of T.

Note that $T \times_k \overline{k} \cong \mathbb{G}^3_{\mathrm{m},\overline{k}}$ and $A \times_k \overline{k} \cong \mu^3_{2,\overline{k}}$.

Theorem (S.)

1) BA is stably rational and $\{BA\} \neq 1$.
2) BT is stably rational and $\{BT\}\{T\} \neq 1$.

Questions 1) and 2) remain open in the case where k is algebraically closed.

(When $k = \overline{k}$ and G is connected, in all known examples BG is stably rational.)

Sketch of proof

Use cut-and-paste methods and short exact sequences to show that $\{BA\} = \{BT\}\{T\}$, and that (1) and (2) are equivalent to an equality of the form

$$(*) \quad (\{K\} - \{E_1\} - \{E_2\} - \{E_3\} + 2)f(\mathbb{L}) = 0 \in K_0(\mathrm{Var}_k),$$

where $f(\mathbb{L})$ is a product of $\mathbb{L}$ and $\mathbb{L}^n - 1$.
Let R be the abelian group generated by classes $[M]$ of finite-dimensional $\mathbb{F}_2$-linear representations M of $\mathrm{Gal}(\overline{k}/k)$, modulo $[M \oplus N] = [M] + [N]$. There is a ring homomorphism $K_0(Var_k) \to R[t]$ such that for all smooth projective X:

$$\{X\} \mapsto \sum_i [H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{F}_2)]t^i.$$

If $\mathrm{char}\, k = 0$, use Bittner presentation (weak factorization). If $\mathrm{char}\, k \neq 2$: $K_0(\mathrm{Var}_k) \to K_0(\mathrm{DM}_{gm}(k; \mathbb{F}_2)) \xrightarrow{\sim} K_0(\mathrm{Chow}(k; \mathbb{F}_2))$, using work of Bondarko. Apply this homomorphism to $(*)$.

Sketch of proof

$$(*) \quad (\{K\} - \{E_1\} - \{E_2\} - \{E_3\} + 2)f(\mathbb{L}) = 0.$$

Let $\Gamma = \mathrm{Gal}(K/k) \simeq (\mathbb{Z}/2)^2$, and $\Gamma_i = \mathrm{Gal}(K/E_i)$. Get:

$$([\mathbb{F}_2[\Gamma]] - \sum_i [\mathbb{F}_2[\Gamma/\Gamma_i]] + 2[\mathbb{F}_2])f(t^2) = 0$$

in $R[t]$. Looking at the leading term in t yields

$$[\mathbb{F}_2[\Gamma] \oplus \mathbb{F}_2^2] = [\oplus_i \mathbb{F}_2[\Gamma/\Gamma_i]].$$

As a $\mathbb{Z}$-module, R is freely generated by $\{M,\ M \text{ indecomposable}\}$, hence

$$\mathbb{F}_2[\Gamma] \oplus \mathbb{F}_2^2 \simeq \oplus_i \mathbb{F}_2[\Gamma/\Gamma_i],$$

which is impossible by Krull-Schmidt Theorem.

Unramified H^3 and integral Hodge Conjecture

Let G be a finite group, and let V be a faithful complex representation of G.

Theorem (Ekedahl)

If $\mathrm{Br}_{nr}(\mathbb{C}(V)^G) \neq 0$, *then* $\{BG\} \neq 1$ *in* $K_0(\mathrm{Stacks}_{\mathbb{C}})$.

Theorem (S.)

If $H^3_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z}) \neq 0$, *then* $\{BG\} \neq 1$ *in* $K_0(\mathrm{Stacks}_{\mathbb{C}})$.

The main ingredient is the work of Colliot-Thélène and Voisin relating H^3_{nr} to the integral Hodge question for cycles of codimension 2.
Combining this with work of Peyre, we find G such that $\{BG\} \neq 1$ and $\mathrm{Br}_{nr}(\mathbb{C}(V)^G) = 0$.

Sketch of Proof

- If X is a smooth projective complex variety and $i \in \mathbb{Z}$, $Z^{2i}(X) := H^{2i}(X(\mathbb{C});\mathbb{Z}) \cap H^{i,i}(X(\mathbb{C});\mathbb{C})/\{\text{alg. classes}\}$, and $Z_{2i}(X) := Z^{2\dim(X)-2i}(X)$.
- $K_0(\mathrm{Ab})$ free abelian group generated by isom. classes of fin. gen. abelian groups, modulo $[M \oplus N] = [M] + [N]$,
- $\hat{K}_0(\mathrm{Var}_{\mathbb{C}})$ completion of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ wrt dimension filtration: $\{Y_m\}/\mathbb{L}^m \to 0$ if $\dim(Y_m) - m \to -\infty$. There are natural maps $K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Stacks}_{\mathbb{C}}) \to \hat{K}_0(\mathrm{Var}_{\mathbb{C}})$.

- $\forall i \in \mathbb{Z}$, $Z_{2i} : \hat{K}_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Ab})$ (additive, cont.) such that if X is smooth projective $Z_{2i}(\{X\}/\mathbb{L}^m) = Z_{2i+2m}(X)$.
- $\{BG\} = \lim_{m\to\infty}\{V^m/G\}\mathbb{L}^{-m\dim(V)}$.

Let $m \gg 0$. There is $X \sim_{\mathrm{bir}} V^m/G$ smooth projective such that $\{V^m/G\} = \{X\} + (\text{smaller dim.})$. We obtain

$$Z_{-4}(\{BG\}) = [Z^4(X)] = [H^3_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})].$$

The second $=$ holds by a theorem of Colliot-Thélène and Voisin.

The Mixed Tate property

A motive $M \in \mathrm{DM}(\mathbb{C};\mathbb{Z})$ is *mixed Tate* if it belongs to the smallest strictly full triangulated subcategory of $\mathrm{DM}(\mathbb{C};\mathbb{Z})$ closed under direct sums in $\mathrm{DM}(\mathbb{C};\mathbb{Z})$ and containing $M^c(\mathbb{A}^n)$ for all $n \geq 0$.

Let G be a finite group. Totaro defined the compactly supported motive $M^c(BG) \in \mathrm{DM}(\mathbb{C};\mathbb{Z})$.

Question (Totaro)

Is is true that $M^c(BG)$ is mixed Tate if and only if $\{BG\} = 1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$?

The Mixed Tate property

For every $i \in \mathbb{Z}$, Ekedahl defined a group homomorphism

$$e_i : K_0(\mathrm{Stacks}_{\mathbb{C}}) \to K_0(\mathrm{Ab}), \quad \{X\}/\mathbb{L}^m \mapsto [H^{i+2m}(X(\mathbb{C}); \mathbb{Z})],$$

where X is smooth and projective. We have $e_0(\{BG\}) = [\mathbb{Z}]$ and $e_2(\{BG\}) = [\mathrm{Br}_{nr}(\mathbb{C}(V)^G)]$. If $\{BG\} = 1$, then $e_i(\{BG\}) = 0$ for all $i \neq 0$.

Theorem (S.)

If $M^c(BG)$ is mixed Tate, then $e_i(\{BG\}) = 0$ for all $i \neq 0$.

Idea: if $M^c(BG)$ is mixed Tate, so is $M^c(\mathrm{GL}_n / G)$, hence $[M^c(\mathrm{GL}_n / G)] \in K_0(\mathrm{DM}_{gm}(\mathbb{C}; \mathbb{Z}))$ is a rational function of $\mathbb{L}$. We have $\{BG\} = \{\mathrm{GL}_n / G\}\{B\,\mathrm{GL}_n\}$. By Bondarko

$$\nu : K_0(\mathrm{Var}_k) \to K_0(\mathrm{DM}_{gm}(\mathbb{C}; \mathbb{Z})) \xrightarrow{\sim} K_0(\mathrm{Chow}(\mathbb{C}; \mathbb{Z}))$$

and so $\nu(\{BG\})$ is a rat. function of $\mathbb{L}$, hence $\nu(\{BG\}) = 1$. Conclude by showing that the e_i factor through ν.

Thank you!

Thank you for your attention!