# Quadratic forms on rings and the homotopy limit problem

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Hermitian K-theory: Witt, Knebusch, Karoubi, Ranicki, Weiss, Williams, Wall, Schlichting, Spitzweck, Lurie, Balmer, ...!

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## § Underived story

$R$ ring, $\mathrm{Proj}_R$ = category of f.g. proj. $R$-modules

comm. monoid

$K_0(R) := \{\text{isoclasses of elements of } \mathrm{Proj}_R, \oplus\}^{gp}$

- $R$ local $K_0(R) \simeq \mathbb{Z}$
- $R$ Dedekind domain $K_0(R) \simeq \mathbb{Z} \oplus \mathrm{Pic}(R)$

<u>Def</u>: $R$ commutative, a <u>symmetric bilinear form</u> is a pair $(P, b)$, $P \in \mathrm{Proj}_R$, $b \in \mathrm{Hom}_R(P \otimes_R P, R)^{C_2}$, i.e. $b: P \times P \to R$ $R$-bilinear s.t. $b(x,y) = b(y,x)$.
$(P, b)$ <u>unimodular</u> if the induced map $P \to P^\vee$ $(x \mapsto b(x, -))$ is an isomorphism.

An isomorphism of symm. bilinear forms, is an iso in $\mathrm{Proj}_R$ sending $b$ to $b$

genuine symmetric

$GW_0^{gs}(R) = \{\text{isoclasses of unimodular symm. bilinear forms}, \oplus\}^{gp}$

<u>Ex</u>: $u \in R^\times$, $\langle u \rangle = (R, b_u)$ $b_u(x,y) = uxy$. $(b \oplus b')(x,x';y,y') = b(x,y) + b'(x';y')$

- $R$ local $GW_0^{gs}(R)$ is generated as an abelian by elements of the form $\langle u \rangle$, $u \in R^\times$.
- $F$ finite field $GW_0^{gs}(F) \simeq \mathbb{Z} \times \mathbb{Z}/2$ $(\simeq \mathbb{Z} \times F^\times/2)$ char $F \neq 2$
- $F$ perf field char 2 $GW_0^{gs}(F) \simeq \mathbb{Z}$
- $GW_0^{gs}(\mathbb{R}) \simeq \mathbb{Z} \times \mathbb{Z}$ $\left(\mathrm{rk}, \frac{\mathrm{rk} - \mathrm{sign}}{2}\right)$ (Sylvester's thm)
- $GW_0^{gs}(\mathbb{Z}) \xrightarrow{\sim} GW_0^{gs}(\mathbb{R})$

We have important maps $GW_0^{gs} \xrightarrow{\mathrm{fgt}} K_0$ forgets the form

$K_0 \xrightarrow{\mathrm{hyp}} GW_0^{gs}$

$P \longmapsto \left(P \oplus P^\vee, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ $b(x,\varphi; x',\varphi') = \varphi x' + \varphi' x$.

finite group of order 2 = $C_2 \circlearrowright K_0$ $P \mapsto P^\vee$

<u>Q1</u>: What's the difference between $GW_0^{gs} \xrightarrow{\mathrm{fgt}} K_0^{C_2}$? (Baby homotopy limit problem)

<u>Def</u>: A quadratic form is a pair $(P, q)$ $P \in \mathrm{Proj}_R$, $q: P \to R$ s.t.

- $q(rx) = r^2 q(x)$
- $b_q(x,y) := q(x+y) - q(x) - q(y)$ is symmetric bilinear

<u>Exercise</u>: $\{\text{Quadratic forms on } P\} \overset{(?)}{\hookrightarrow} \mathrm{Hom}_R(P \otimes_R P, R)_{C_2}$

$x \mapsto b(x,x)$ $\longleftarrow\!\shortmid$ $b$

<u>Def</u>: $q$ is unimodular iff $b_q$ is.

$GW_0^{gq}(R) = \{\text{unimodular quadratic forms}\}^{gp}$

One can do antisymm, hermitian (if $R$ has an involution), $(-1)$-quadratic, alternating, even, ...

<u>Def</u>: A form functor on $R$ is a functor

$\Phi: \mathrm{Proj}_R^{op} \to \mathrm{Ab}$

① $\Phi(0) = 0$

② $\Phi(P \oplus Q) \simeq \Phi(P) \oplus \Phi(Q) \oplus B_\Phi(P, Q)$

$$\begin{tikzcd} Q \arrow[r] \arrow[rr, bend left, equal] & P \oplus Q \arrow[r] \arrow[d] & Q \\ P \arrow[ur] \arrow[rr, bend right, equal] & & P \end{tikzcd}$$

$B_\Phi(P, Q) \simeq \mathrm{Hom}_R(P, DQ)$ for $D: \mathrm{Proj}_R^{op} \to \mathrm{Proj}_R$ <u>duality</u>

$B_\Phi(P,Q) \simeq B_\Phi(Q,P) \Rightarrow \eta: 1 \xrightarrow{\sim} DD^{op}$

<u>Ex</u>: $\Phi(P)$ = abelian group of
- symm bil forms on $P$ } same duality
- quadratic forms
- antisymm bil forms — different duality: same functor $P \mapsto P^\vee$ different double dual ident. $P \xrightarrow{-1} (P^\vee)^\vee$

$\Phi(P) \xrightarrow{\nabla^*} \Phi(P \oplus P) \simeq \Phi(P) \oplus \Phi(P) \oplus \mathrm{Hom}(P, DP) \to \mathrm{Hom}(P, DP)$

$q \longmapsto q_\# : P \to DP$

<u>Def</u>: $q \in \Phi(P)$ unimodular, $q_\#$ is an isom.

$GW_0(R, \Phi) = \{\Phi\text{-quadr. forms}\}^{gp}$.

<u>Ex</u>: $GW_0^{gq}(\mathbb{Z}) \hookrightarrow GW_0^{gs}(\mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}$ $\mathrm{sign} \equiv 0\ (8)$

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Space = $\infty$-cat. of spaces, $\infty$-cat. of animas/anima

objects: (weak) homotopy types, $\forall X, Y \in$ Space $\mathrm{Map}(X, Y)$ homotopy type "mapping space"

$[K, \mathrm{Map}(X,Y)]$ "functions $K \times X \to Y$ up to htpy".

$\mathrm{Gpd} \to \mathrm{Space}$

$\mathcal{X} \longmapsto \coprod_x B\mathrm{Aut}_{\mathcal{X}}(x) = |\mathcal{X}|$

"Embedding is fully faithful" $\mathrm{Map}(|\mathcal{X}|, |\mathcal{Y}|) \simeq |\mathrm{Fun}(\mathcal{X}, \mathcal{Y})|$

(essential image are htpy types $X$ w/ $\pi_i X = 0$ $\forall i > 1$, i.e. 1-types)

In Space there's a good notion of comm. monoids, & abelian groups which are called $E_\infty$-spaces & $E_\infty$-groups respectively, $E_\infty$-gps $\subseteq E_\infty$-spaces has a left adjoint $X \mapsto X^{gp}$.

$\mathrm{Gpd} \to \mathrm{Space}$ respects this structure, i.e. it sends symmetric monoidal groupoids to $E_\infty$-spaces.

$K(R) := (|\iota \mathrm{Proj}_R|)^{gp}$ $GW(R, \Phi) = (|\text{gpd of unimod } \Phi\text{-quadr. forms}|)^{gp}$

groupoid of proj $R$-modules

⚠ $(-)^{gp}$ preserves discrete objects, but not 1-types

Just by functoriality we get $GW(R, \Phi) \xrightarrow{\mathrm{fgt}} K(R)$, $K(R) \xrightarrow{\mathrm{hyp}} GW(R, \Phi)$

& a $C_2$-action $C_2 \circlearrowright K(R, \Phi)$

fixed pts in Space, $E_\infty$-groups

<u>Q</u>: $GW(R, \Phi^{gs}) \xrightarrow{\sim} K(R, \Phi)^{hC_2}$, after 2-completion? (Homotopy limit problem)

<u>A</u>: True: whenever $\mathrm{vcd}_2(R) := \sup\left(\mathrm{cd}_2(k(p)[\sqrt{-1}]) \mid p \in \mathrm{Spec}\, R\right) < \infty$

$\dim R < \infty$

$\frac{1}{2} \in R$

(Hu-Kriz-Ormsby $R$ field, Karoubi-Schlichting-Østvær)

- True for perfect fields of characteristic 2 (9 authors, HKT3)
- True for orders in global fields of char $\neq 2$ (9 authors, HKT3)
- True for fields $F$ of char 2 s.t. $[F : F^2] < \infty$ (N., unpublished)
- True for orders in all global fields (N., unpublished)

$R$ non dyadic hens. local ring $GW(R)^\wedge_2 \xrightarrow{\sim} GW(k)^\wedge_2$, $K(R)^\wedge_2 \xrightarrow{\sim} K(k)^\wedge_2$

Balmer, Knebusch

Suslin

- You can assemble other unpublished results by various combinations of 9 authors, J. Shah, I. Patchkoria, ... to get more cases: perfect $\mathbb{F}_2$-algebras.

You can use these results to completely compute $GW_*^{gs}(F)$ $F$ field of char 2

$$\pi_n GW^{gs}(F) = \begin{cases} 0 & n \text{ odd} \\ K_n(F) \times_{K_n(F)/2} I(F)^n \end{cases}$$

ker $(GW_0(F) \to K_0(F))$

$K_{hC_2} \to GW \to L$

$$|\mathrm{Cob}_d| \to |\mathrm{Cob}_d(R)| \simeq \underline{\underline{\Omega^{\infty+1} GW^s(R)}}$$

<u>Homotopy limit problem</u>

Fields: motivic methods (motivic ss for K-theory: Bloch-Kato + ...)

Fields $\rightsquigarrow$ henselian local rings: trace methods $\left(GW^q(R) \to GW^{gs}(R) \to \underline{TCR}(R)^{\phi C_2}\right)$

+ perfectoid methods Harpaz - Nikolaus - Shah

Dotto, Patchkoria, Moi, Rüdiger

henselian local $\rightsquigarrow$ global: descent argument (9 authors, Calmès - Harpaz - N., Schlichting ($\frac{1}{2} \in R$))