Constructing Logarithmic Moduli

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STANFORD AG
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CURVE COUNTING:

GROMOV-WITTEN

DONALDSON-THOMAS

Stable Maps(X) — MNOP — Hilbert Scheme(X)

Give X a SNC divisor $\boxed{D \subseteq X}$.

Logarithmic Maps$(X|D)$ — ...?

TODAY

ABRAMOVICH – CHEN

GROSS – SIEBERT

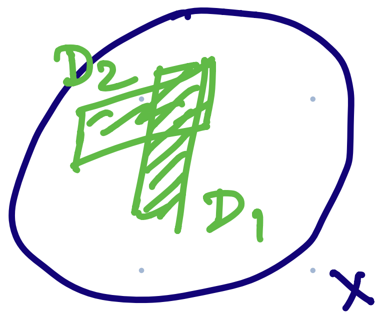
- J. Li & Li-Wu

- why is it interesting?
- why is it different?

Logarithmic Hilbert Scheme of Curves.

X : smooth proj. variety

Fix $\boxed{D \subseteq X \quad \text{SNC}}$. Start with the usual



Hilbert scheme:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & X \times \mathrm{Hilb}(X) \arrow[d] \\ & p \in \mathrm{Hilb}(X) \end{tikzcd}$$

The divisor $D \subseteq X$ determines an interesting locus:

$$\begin{tikzcd} Z_p \arrow[r, hook] \arrow[rr, bend right, "\text{FLAT}"'] & X \arrow[r, "\text{``}D\text{''}"] & {[\mathbb{A}^r / \mathbb{G}_m^r]} \end{tikzcd}$$

The transverse locus:

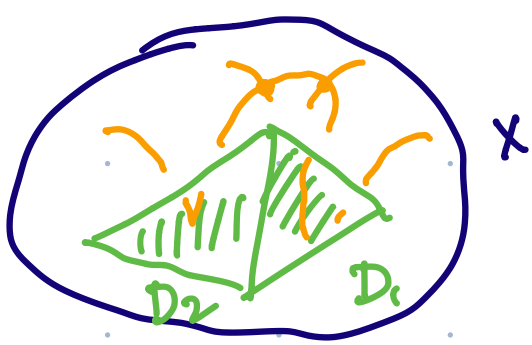
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[rrr, bend left] & \mathrm{Hilb}^\circ(X|D) \times X \arrow[r] \arrow[dl] & X \arrow[r] & {[\mathbb{A}^r / \mathbb{G}_m^r]} \\ \mathrm{Hilb}^\circ(X|D) & & & \end{tikzcd}$$

Geometric Consequence: If $D_i \subseteq D$ is a component

X: 3-fold

$$ev_i : \mathrm{Hilb}^\circ(X|D) \longrightarrow \mathrm{Hilb}^\circ(D_i \,|\, D_i \cap D)$$

curves points.



The <u>ideal</u> outcome:

Logarithmic Hilbert schemes:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[r] \arrow[dl] & X \\
\mathrm{Hilb}^{\log}_{\text{curves}}(X|D) \arrow[rr, "ev_i"'] & & \mathrm{Hilb}^{\log}_{\text{pts}}(D_i|D_i \cap D)
\end{tikzcd}$$

<u>FIRST APPROXIMATION</u>:

Study subschemes in <u>logarithmic expansions</u> of X along D.

— I want to explain how to build this. —

Torus Orbit Closures: in Hilbert schemes.

Say $Z^0 \hookrightarrow \mathbb{G}_m^r$

what is a good toric compactification $X \supseteq \mathbb{G}_m^r$ in which to compactify Z^0?

"Good" here:

$$\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r] & {[X/\mathbb{G}_m^r]} \\ Z^0 \arrow[u, hook] \arrow[r, hook] & \mathbb{G}_m^r \arrow[u, hook] & \end{tikzcd}$$

— Flat

- Does it exist?
- How to construct it?

Tevelev (Kapranov, Ulirsch, Gubler, ...)

1. Using Z° build: $\mathbb{G}_m^r \hookrightarrow \mathrm{Hilb}(\mathbb{P}^r)$

2. Take the closure of $\mathbb{G}_m^r$ & normalize to get Y.

THEOREM: The closure of Z° in Y is good

$$\begin{tikzcd} \overline{Z^\circ} \arrow[r, hook] \arrow[rr, bend right, "\text{FLAT.}"'] & Y \arrow[r] & {[Y/\mathbb{G}_m^r]} \end{tikzcd}$$

$\boxed{\Sigma_Y \text{ on } N_{\mathbb{R}}.}$

What does it look like? Tropicalization!

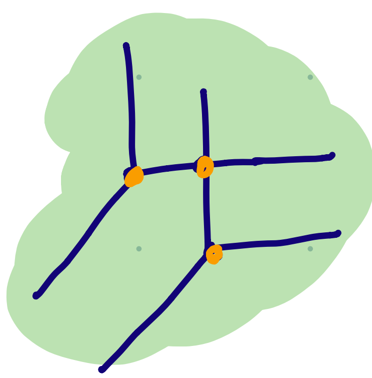
Extend scalars to $K = \mathbb{C}((t^{\mathbb{R}}))$ absurdly large field with valuation.

Compute image of Z° under:

$$\mathrm{trop} : \mathbb{G}_m^r(K) \longrightarrow \mathbb{R}^r$$
$$(z_1, \dots, z_r) \longmapsto (\mathrm{val}(z_1), \dots, \mathrm{val}(z_r))$$

$\mathrm{trop}(Z^\circ)$ is the support of a polyhedral complex

Σ_Y is supported on $\mathrm{trop}(Z^\circ)$.

[This is where these pictures  come from]

How is this relevant?

If you squint right...

SNC PAIR: $D \subseteq X$; r smooth divisors $D_1, \ldots, D_r$ meeting locally like coordinate planes in $\mathbb{A}^r$.

$(X|D) \times \Delta$

$\hookrightarrow$ SNC pair

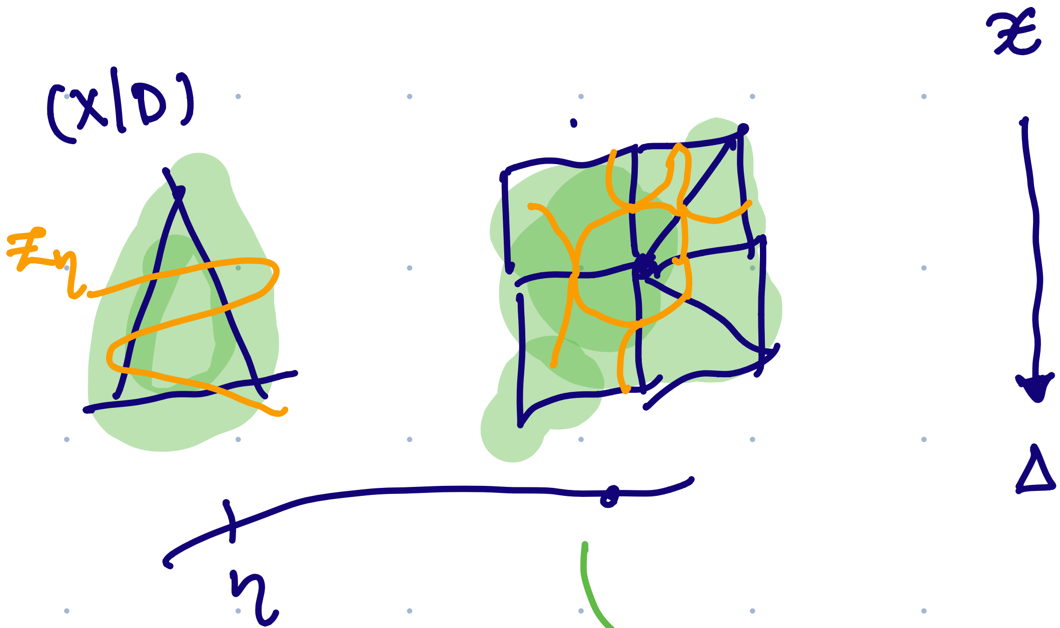
Input from $\mathrm{Hilb}^\circ(X|D)$

$$\begin{tikzcd} Z_{\Delta^\circ} \arrow[r, hook] \arrow[dr] & (X|D) \arrow[d] \\ & \Delta^\circ \end{tikzcd}$$

see Ulirsch's first paper.

... a family of transverse subschemes

Output is a completion of the family:



AN "EXPANSION"

The Universal Expansion

We want a moduli stack

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d] & X \\ \mathrm{Exp}(X|D) & \end{tikzcd}$$

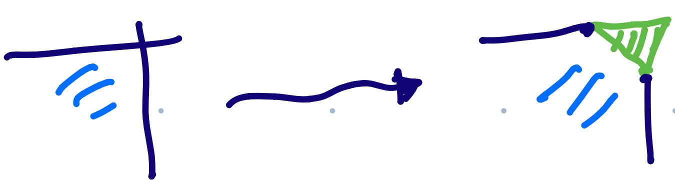
How do we build an expansion?

$$\begin{tikzcd} (X|D) \arrow[r, squiggly, "\text{FAN}"] & \Sigma_{X|D} \arrow[r, squiggly, "\text{TRIVIAL}", "\text{FAMILY}"'] & \Sigma_{X|D} \times \mathbb{R}_{\geq 0} \arrow[d] \\ & & \mathbb{R}_{\geq 0}. \end{tikzcd}$$

$X \times \mathbb{A}^1 \to \mathbb{A}^1$

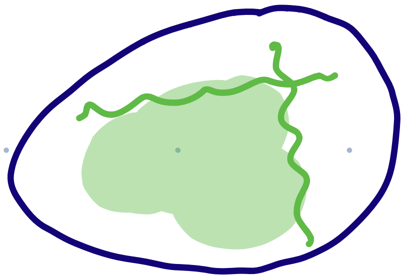
$$\begin{tikzcd} {} \arrow[r, squiggly, "\text{SUBDIVIDE}"] & \widetilde{\Sigma_{X|D} \times \mathbb{R}_{\geq 0}} \arrow[d] \arrow[r, squiggly, "\text{TORIC}", "\text{DICTIONARY}"'] & \mathfrak{X} \arrow[r] & \mathbb{A}^1 \\ & \mathbb{R}_{\geq 0} & \mathfrak{X}_0. \arrow[u, squiggly] & \end{tikzcd}$$

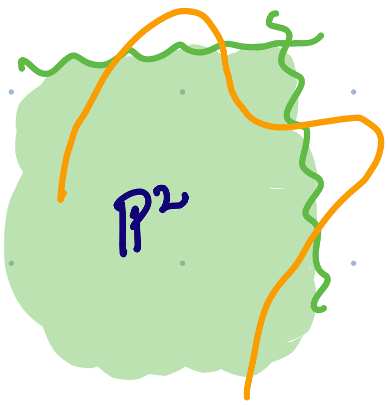
Examples:



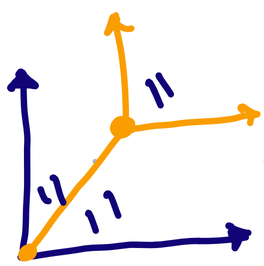
An Expansion

$(\mathbb{P}^2 | L_1 + L_2)$

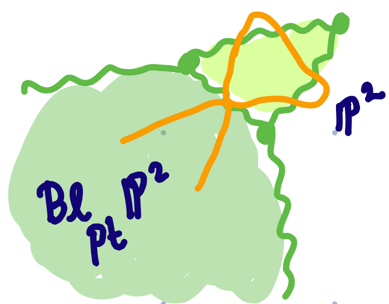




DEGENERATES



DEFORMATION TO THE NORMAL CONE OF $L_1 \cap L_2$



In general:
- Draw a graph in $\Sigma_{X|D}$
- Take a cone over it & apply the toric dictionary

A Strategy

Build the combinatorial moduli space

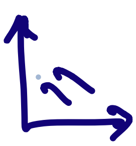
$$\begin{array}{ccc} \tilde{\Sigma} & \longrightarrow & \Sigma \\ \downarrow & & \\ T(\Sigma_{X|D}). & & \end{array}$$

Hopefully it's given by cones glued along faces.

Convert this into an Artin stack [Abramovich-Wise, Chan-Cavalieri-Ulirsch-Wise]

Hopefully end up with $\mathrm{Exp}(X|D)$.

$$\varinjlim \sigma \rightsquigarrow \varinjlim [U_\sigma / T_\sigma]$$

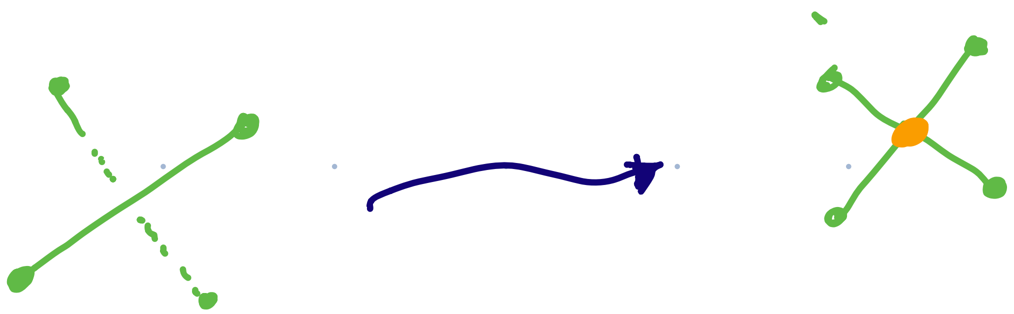


$$\mathbb{R}^2_{\geqslant 0} \leftrightsquigarrow [\mathbb{A}^2 / \mathbb{G}_m^2]$$

The <u>trouble with</u> hope:

$T(\Sigma_{\times ID})$ is not actually glued from cone complexes (not "representable").

<u>Ex</u>: $T(\Sigma_{\times ID})$ should contain a moduli space of embedded metric graphs in $\mathbb{R}^3_{\geq 0}$.



<u>Practically</u>: No minimal polyhedral structure on a polyhedral set.

It works out in the end:

There is an infinite collection

$$\left\{ \mathrm{Exp}(X|D)_\lambda \right\}_\lambda$$

choice of a polyhedral structure on $T(\Sigma_{X|D})$

of birational stacks, each good enough for a logarithmic Hilbert scheme to exist over it.

$$\left\{ \mathrm{Hilb}^{\log}_{\text{curves}}(X|D) \right\}$$

$$\Big\downarrow \mathrm{ev}_i$$

$$\left\{ \mathrm{Hilb}^{\log}_{\text{pts}}(D_i | D_i \cap D) \right\}$$

"The logarithmic Hilbert scheme exists."
(at least for curves)

We relearn lessons: Nature wants us to invert a class of morphisms: $\boxed{\text{SUBDIVISIONS}}$ but calculations in AG don't always allow this.

Similar: $\mathbb{G}_m^{\log}$, $\mathrm{Pic}^{\log}$, ...

cf Raynaud's "Admissible formal schemes"

THEOREM: There exist moduli spaces

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[r] \arrow[dl] & X \\ \mathrm{Hilb}^{\log}_{\text{curves}}(X|D) \arrow[r] & \mathrm{Hilb}^{\log}_{\text{pts}}(D_i | D_i \cap D) \end{tikzcd}$$

with $\mathcal{Z}_p \hookrightarrow \mathfrak{X}_p$ transverse. It is proper, has a virtual class when $\dim X = 3$.

WHY WE STARTED ...

GROMOV-WITTEN THEORY

MNOP

DONALDSON-THOMAS THEORY

LI

GROSS SIEBERT

ABRAMOVICH-CHEN

[& the SYMPLECTOMORPHS]

Li-Wu

Maulik-R '20

[maybe some of you!]

LOGARITHMIC GW THEORY

NEW CONJECTURES

DEGENERATION FORMULAE

etc

LOGARITHMIC DT THEORY

CONNECTIONS

- Logarithmic GW theory redone, $(\mathrm{GW}/\mathrm{DT})^{\mathrm{LOG}}$ conjecture
- Gelfand–Kapranov–Zelevinsky's SECONDARY POLYTOPES.
- Lafforgue's compactifications of thin Schubert cells
- Hacking–Keel–Tevelev: moduli of del Pezzo's & hyperplane arrangements.

& there is a lot to do still !

THANKS !