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Constructing varieties with prescribed Hodge numbers modulo m in positive characteristic

(joint with Matthias Paulsen)

I Introduction

Let X be a smooth projective variety over an algebraically closed field k.

<u>Def</u> The Hodge numbers of X are

$$h^{p,q}(X) = \dim H^q(X, \Omega_X^p).$$

The Hodge diamond is

$$h(X) = \begin{array}{ccccc} & & h^{n,n} & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & \\ h^{n,0} & & \cdots & & h^{0,n} \\ & \ddots & & \cdot^{\cdot^{\cdot}} & \\ & & h^{0,0} & & \end{array}$$

<u>Ex</u>

- $h(C) = \begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$

- $h(\mathbb{P}^2) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$

- $h(\mathbb{P}^n) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ \cdot^{\cdot^{\cdot}} & & & \vdots & & & \ddots \\ 0 & & & & & & 0 \\ \ddots & & & \vdots & & & \cdot^{\cdot^{\cdot}} \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$

(Hartshorne, Exc III. 7.3)

<u>Rmk</u> Relations:

(i) $h^{0,0}(X) = 1$

(2) (Serre duality)

$$h^{p,q}(X) = h^{n-p,n-q}(X)$$

(use SD + perfect pairing $\Omega_X^p \times \Omega_X^{n-p} \to \omega_X$.)

(3) (Hodge symmetry)

If char $k = 0$, then $h^{p,q}(X) = h^{q,p}(X)$.

<u>Thm</u> (Serre, 1958)

If char $k > 0$, there exists a surface S with

$$h(S) = \begin{matrix} & & 1 & & \\ & 1 & & 0 & \\ a & & b & & a \\ & 0 & & 1 & \\ & & 1 & & \end{matrix}.$$

<u>Idea</u> Take a $\mathbb{Z}/p$-quotient $Y \hookrightarrow \mathbb{P}^3 \circlearrowleft \mathbb{Z}/p$.

equivariant hypersurface on which $\mathbb{Z}/p$ acts <u>freely</u>.

$\downarrow \pi$ $\mathbb{Z}/p$-quotient $\Rightarrow$ étale

Then $h(Y) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ \cdot & & \cdot & & \cdot \\ & 0 & & 0 & \\ & & 1 & & \end{array}$, and

$$H^0(S,\Omega^1_S) \xhookrightarrow{\pi^*} H^0(Y,\Omega^1_Y) = 0.$$

recycle : $\mathbb{Z}/p \to \mathbb{G}_m$

But $\underline{\mathrm{Pic}}^0_S = (\mathbb{Z}/p)^\vee = \mu_p$, so

$$H^1(S,\mathcal{O}_S) = T_0\,\underline{\mathrm{Pic}}^0_S \neq 0. \qquad \square$$

since $\mu_p = \mathrm{Spec}\,\dfrac{k[x]}{(x^p-1)} = \mathrm{Spec}\,\dfrac{k[x]}{(x-1)^p}$.

II Inverse Hodge problem

Q Which Hodge diamonds can be realised?

This is very hard.

Ex

- $$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} + \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \times \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$ ✓

$$= h(E) \times h(\mathbb{P}^1) = h(E \times \mathbb{P}^1).$$

- $$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$ cannot be realised: suppose it is $h(X)$.

Then the Albanese map $X \xrightarrow{\alpha} \underline{Alb}_X$ fibres X over an elliptic curve: $X \xrightarrow{\alpha} E$. But then $\alpha^* \mathcal{O}_E(1)$ and $\mathcal{O}_\alpha(1)$ are linearly independent in $NS(X)$, so

$$h^{1,1}(X) \geq 2. \quad ↯ \quad \square$$

$$\sum_{p+q=i} h^{p,q}(X) \geq h^i_{dR}(X) \geq h^i_{crys}(X)$$

Easier question:

Q: What are the linear/polynomial relations satisfied by all Hodge diamonds of smooth projective varieties of dimension n?

I.e. if $(\lambda_{i,j})_{i,j=0}^{n}$ s.t. $\sum \lambda_{i,j} h^{i,j}(X) = 0$ for all $X \in \underline{\mathrm{SmPr}}$, then what can we say about $(\lambda_{i,j})$?

And what is the Zariski closure of $\{h(X) \mid X \in \underline{\mathrm{SmPr}}\} \subseteq \mathbb{A}^{(n+1)^2}$?

$h(X) = \{h^{i,j}(X)\}_{i,j=0}^{n}$.

Thm (Kotschick–Schreieder, 2013)
If char $k = 0$, the only linear relations are (spanned by) SD & HS.

Thm (vDdB, ..., 2021)
If char $k > 0$, the only linear relations are (spanned by) SD,

Thm (Paulsen–Schreieder, 2019) ... (generated by)

Thm (...) ... the only polynomial relations are (generated by)
If char $k = 0$, the only polynomial relations are (generated by)
$h^{0,0} = 1$, SD, and HS.

Thm (vDdB - Paulsen, 2020)
If char $k > 0$, the only polynomial relations are (generated by)
$h^{0,0} = 1$ and SD.

Idea Construct many Hodge diamonds as

(1) linear combinations of Hodge diamonds of varieties
(KS13, vDdB21)

(2) modulo m reductions of Hodge diamonds of varieties
(PS19, vDdBP20).

Thm (KS13, vDdB21)
Given a Hodge diamond $h \in \mathbb{Z}^{(n+1)^2}$ satisfying SD (& HS),

Given a ... $h = a_1 h(X_1) + \dots + a_r h(X_r)$.
there exist $X_1, \dots, X_r \in \underline{\mathrm{SmPr}}$ s.t.
$a_1, \dots, a_r \in \mathbb{Z}$

Thm [PS19, vDdBP20]
Given a Hodge diamond $h \in (\mathbb{Z}/m)^{(n+1)^2}$ satisfying $h^{0,0} = 1$, SD (& HS), there exists $X \in \underline{\mathrm{SmPr}}$ s.t. $h \equiv h(X) \mod m$.

Rmk This implies the theorem above: a Zariski closed subset $Z \subseteq \mathbb{A}^N$ with $Z(\mathbb{Z}/m) = \mathbb{A}^N(\mathbb{Z}/m)$ for all m has to equal $\mathbb{A}^N$.

Ex (1)

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} + \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} - \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$h(E \times \mathbb{P}^1) + h(\mathbb{P}^2) - h(\mathbb{P}^1 \times \mathbb{P}^1)$$

... $(\mathbb{P}^2)$

$$= h(E \times \mathbb{P} \ldots \qquad \text{or } Bl_{pt}(\mathbb{P} \ldots)$$

$$(2) \quad \begin{matrix} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix} \equiv \begin{matrix} & & 1 & & \\ & 1 & & 1 & \\ 0 & & m+1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix} = h\left(Bl_{(m-1)pts}(E \times \mathbb{P}^1)\right) \bmod m$$

III Sketch of proof (vDdBP20)

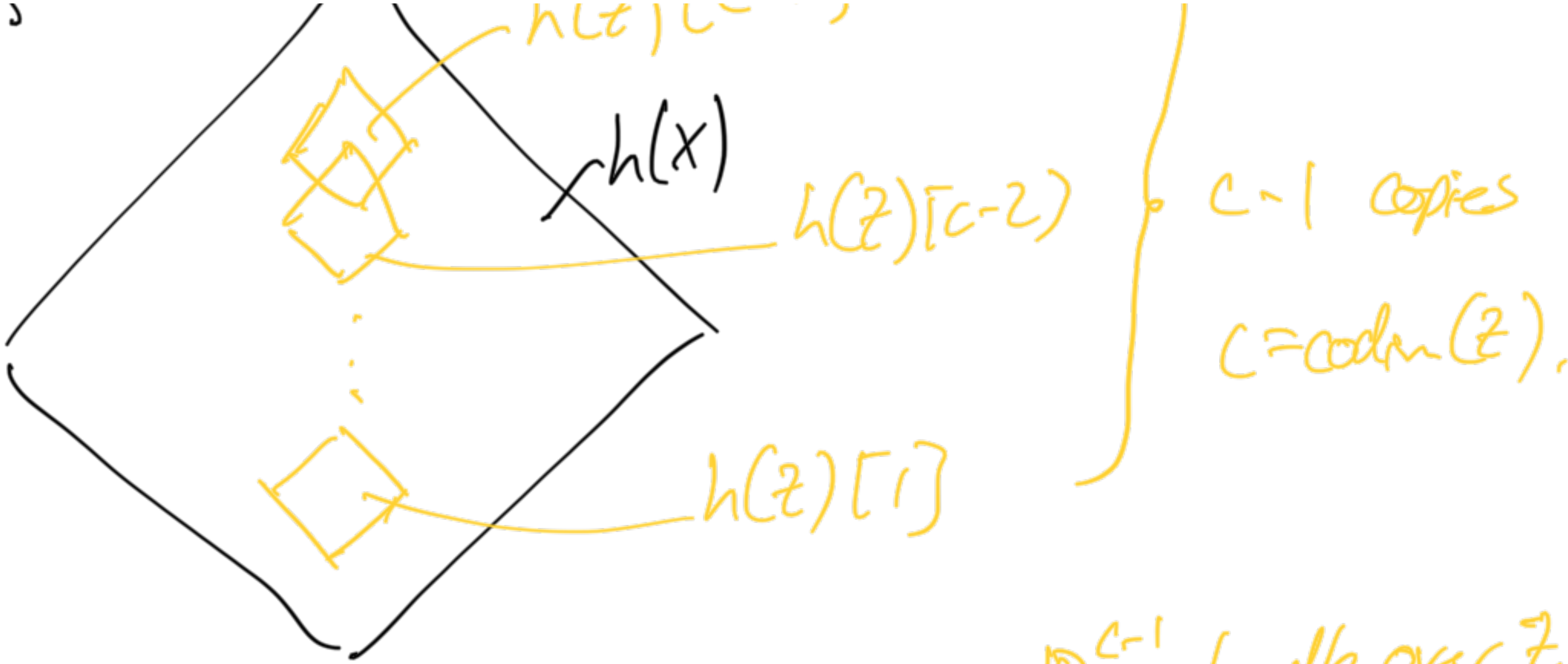
Some tools:

(K) Künneth: $\begin{matrix} & 1 & \\ a & & a \\ & 1 & \end{matrix} \times \begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix} = \begin{matrix} & & 1 & & \\ & a & & a & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{matrix} + \begin{matrix} & & 0 & & \\ & 1 & & 0 & \\ a & & a & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{matrix} + \begin{matrix} & & 0 & & \\ & 0 & & 1 & \\ 0 & & a & & a \\ & 0 & & 1 & \\ & & 0 & & \end{matrix} + \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & a & & a & \\ & & 1 & & \end{matrix}$

$$= \begin{matrix} & & 1 & & \\ & a+1 & & a+1 & \\ a & & 2a+2 & & a \\ & a+1 & & a+1 & \\ & & 1 & & \end{matrix}.$$

(B) blowups:

$h(\mathrm{Bl}_Z(X))$



$\mathbb{P}^{c-1}$-bundle over Z

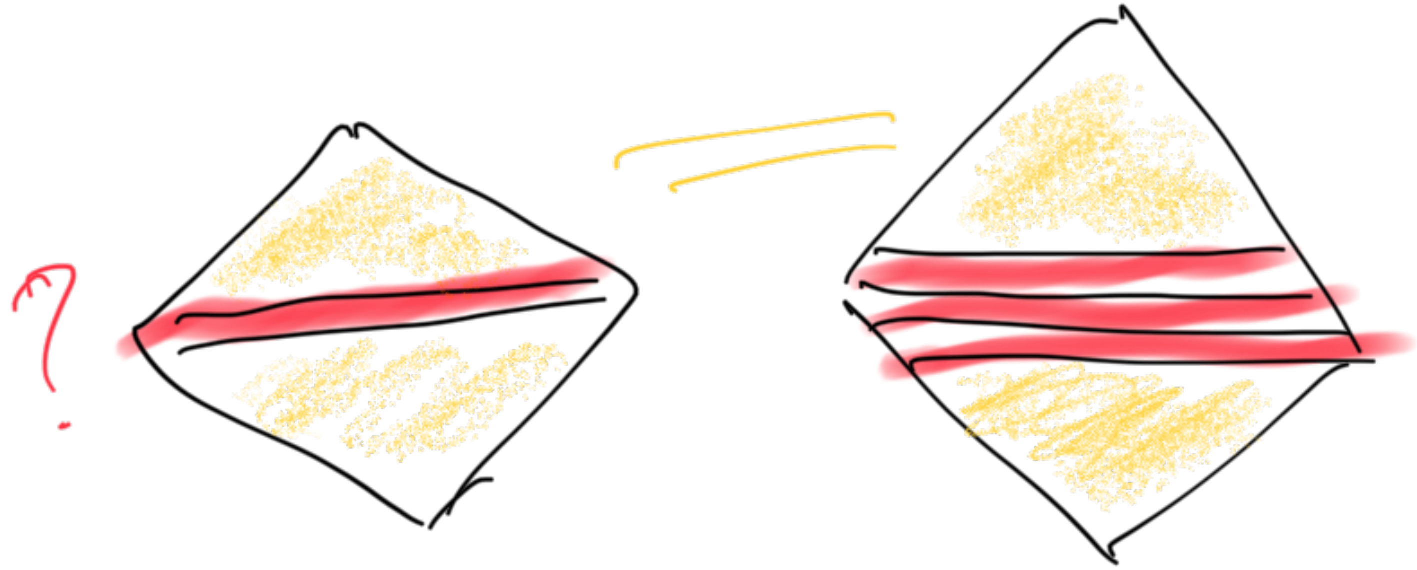
Formula:

$$h(\mathrm{Bl}_Z X) = h(X) - h(Z) + h(E)$$

$$= \underline{h(X)} - \cancel{h(Z)} + h(Z)(\cancel{1} + \underline{\mathbb{L}} + \underline{\mathbb{L}^2} + \dots + \underline{\mathbb{L}^{c-1}})$$

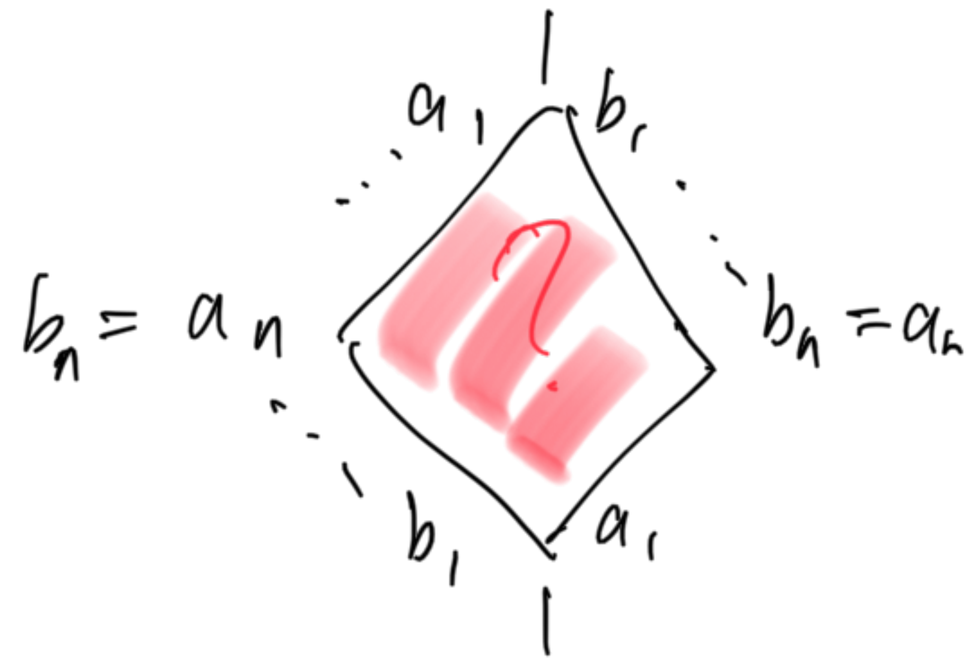
(C) weak Lefschetz: if $D \subseteq X$ is sufficiently ample, then

$$h(D) \subseteq h(X)$$



Steps in proof:

(1) By induction (using K + WL) construct any outer edge



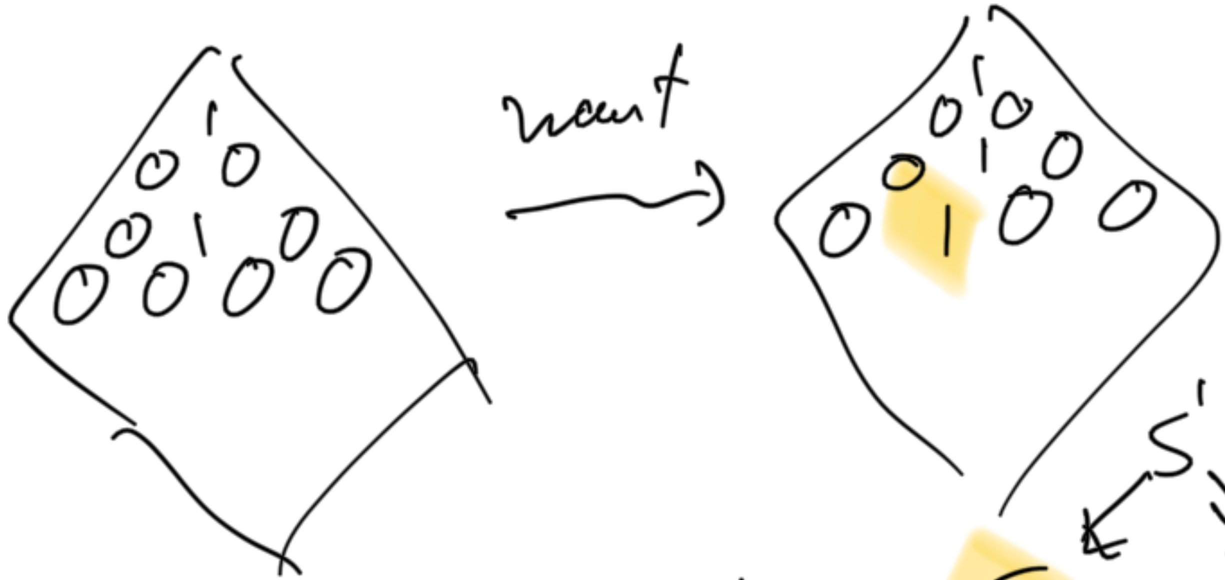
(2) Get the correct inner Hodge numbers by blowup

(2) ... constructions:

(2a) Produce subvarieties with nice Hodge numbers by blowing up. ⇝ Do the same construction m times, you don't change $h(X)$ mod m.

given

dim ≥ 4



This is of one X contains S

$$\begin{array}{ccc} & S' & \\ \swarrow & & \searrow \\ S & & \bar{S} \end{array}$$

$\mathbb{P}^{n-1} \supseteq \mathbb{P}^{N+1} \supseteq CY^N$

(i) Blow up a pt to get $\mathbb{P}^3$

(ii) $\bar{S} \subseteq \mathbb{P}^3$ birational to S

(iii) embedded resolution of $\bar{S} \subseteq \mathbb{P}^3$.

(2b) Blow up one of these subvarieties (in controlled way).

inductions, ...

$\square$