PATHOLOGIES ON THE HILBERT SCHEME OF POINTS

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(local time 21:00, weather: mostly sunny)

Includes 3+1 open questions.

COMMENTS WELCOME!

NOTATION

$\mathrm{HILB}^{p}(\mathbb{P}^n) = \{[Z] \mid Z \subseteq \mathbb{P}^n,$ Hilb poly of Z is $p\}$. $p \in \mathbb{Q}[t]$

$\mathrm{Hilb}_d(\mathbb{P}^n) := \mathrm{Hilb}^d(\mathbb{P}^n)$. ALSO $\mathrm{Hilb}_d(\mathbb{A}^n)$ $\quad d \in \mathbb{Z}_{>0}$

VERY INCOMPLETE HISTORY

WHO?

1961 GROTHENDIECK — construction of $\mathrm{Hilb}^p(\mathbb{P}^n)$

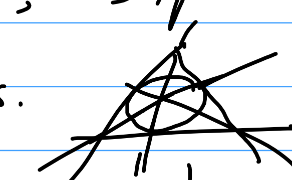
1962 MUMFORD — a generically nonreduced component of $\mathrm{Hilb}^p(\mathbb{P}^3)$ $\deg p = 1$

1966 HARTSHORNE — connectedness $\mathbb{P}^n$! $\forall p$

1968 FOGARTY — $\mathrm{Hilb}^p(\mathbb{P}^2)$ smooth $\forall p$ $\quad \mathrm{Hilb}^d(\mathbb{P}^1) = \mathbb{P}^d$.

1972 IARROBINO — $\mathrm{Hilb}_d(\mathbb{P}^3)$ reducible $\Rightarrow$ singular $\quad \mathbb{P}^2$, lines, pts

1985-89 (+2012) MNËV - STURMFELS, LEE - VAKIL — arbitrary singularities on configuration spaces.



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2001 CIOCAN-FONTANINE KAPRANOV — $\mathrm{Hilb}^p(\mathbb{P}^n)$ admits a derived-smooth

2005 VAKIL — arbitrary singularities on $\mathrm{Hilb}^p(\mathbb{P}^n)$ for any $\deg p$

at $[Z]$ $\quad Z$ smooth

$X \longrightarrow \operatorname{Spec} \mathbb{Z}$ finite type.

SINGULARITY TYPES

$$(X,x) \sim (Y,y) \qquad (1)$$

$$\Updownarrow$$

$$\hat{\mathcal{O}}_{X,x}[[t_1, \dots, t_n]] \cong \hat{\mathcal{O}}_{Y,y}[[u_1, \dots, u_m]]$$

$(1) \Longleftrightarrow (2)$ eq Artin

$$(2)$$

$$(X,x) \sim (Y,y)$$

$$\Updownarrow$$

$$\begin{tikzcd} & (U,u) \arrow[dl, "\text{smooth}"'] \arrow[dr, "\text{smooth}"] & \\ (X,x) & & (Y,y) \end{tikzcd}$$

Def A SING. TYPE is a class:

$$[(X,x)] \text{ under } \sim.$$

Ex:

- $[\operatorname{Spec} \mathbb{F}_p] \ni (X,x) \iff \hat{\mathcal{O}}_{X,x}$ char p & regular.
- $\left[\operatorname{Spec} \frac{\mathbb{Z}[x]}{x^2}\right] \ni (Y,y) \implies y \in Y$ nonreduced.

$$\neq \left[\operatorname{Spec} \frac{\mathbb{Z}[y]}{y^3}\right]$$

$\forall d > 0$

Thm (VAKIL): $\forall \Sigma$ sing. type

$\exists\ \exists [Z] \in \mathrm{Hilb}^p(\mathbb{P}^n)$

deg $p = d$

$[\mathrm{Hilb}^p(\mathbb{P}^n), [Z]] \in \Sigma$

Z smooth

Proof: deg $p = 1$.

Fantechi–Pardini: $S \subseteq \mathbb{P}^n$ smooth surface

$H \subseteq \mathbb{P}^n$ hypersurface, deg $H \gg 0$

$\cup$

$\cap$

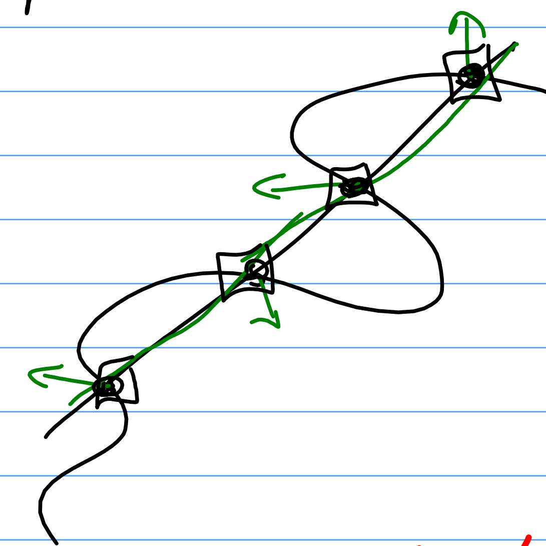
$\mathrm{Hilb} \times \mathrm{Hilb} \overset{\pi}{\dashrightarrow} \mathrm{Hilb}$

$[S], [H] \longmapsto [S \cap H]$

π is smooth near $[S \cap H]$

If $(\mathrm{Hilb}, [S]) \in \Sigma \Rightarrow (\mathrm{Hilb}, [S \cap H]) \in \Sigma$

This step fails for S a curve.



OPEN PROBLEM deg $p = 1$

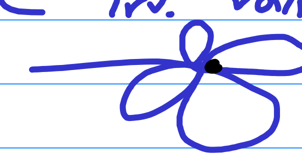
$C \boxed{\subseteq \mathbb{P}^n}$ $\quad (\mathrm{Hilb}, [C]) \in \Sigma$

smooth

$H^2(T_C) = 0$ $\quad \mathrm{Def}_C$ are unobstructed

FIND integral C non-normal $/ \mathbb{F}_p$ $\quad C$ does not lift to char 0?

- Can assume C irr. rational single singular pt



(Q: $\mathrm{Hilb}_{21}(\mathbb{A}^n)$ has gen. nonred. comps. $n \gg 2$)

Thm $\coprod_d \mathrm{Hilb}_d(\mathbb{A}^{16})$ has all singularity types ~~up to retraction~~.
(—)

In particular:

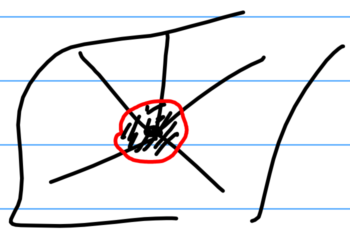
- nonreduced
- components in char p $\forall p$.

ERMAN'12 $V(I_S) = \hat{S} \subseteq \mathbb{A}^{n+1}$

$S \subseteq \mathbb{P}^n$, $I_S \subseteq k[x_0 \ldots x_n]$

$$\downarrow$$

$$(I_S)_{\geq e} + \underline{(x_0 \ldots x_n)^{e+2}} =: J$$



$e > \mathrm{reg}(I_S) + 2$

$$\mathrm{Def}_{S \subseteq \mathbb{P}^n} \simeq \mathrm{Def}(I_S)_{\geq e}^{\mathbb{G}_m} \simeq (\mathrm{Def}\, J)^{\mathbb{G}_m}$$

MORAL: $\mathbb{G}_m$-equivariant gives $\mathcal{O}(1)$.

$\underline{V(J)}$ zero-dimensional $\Rightarrow$ line bdls TRIVIAL

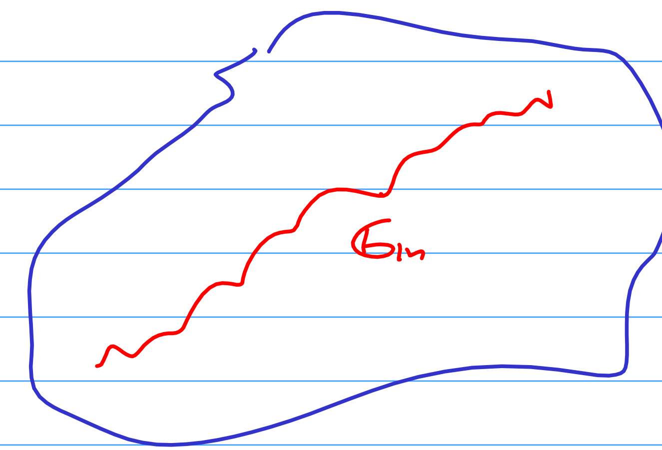
Thm (Erman): $\coprod_{n,d} \mathrm{Hilb}_d(\mathbb{A}^n)^{\boxed{\mathbb{G}_m}}$ has all singularity type

$H := \mathrm{Hilb}$, ignore $\coprod_{n,d}$, $\coprod_d$

$$H_d(\mathbb{A}^n)^{\mathbb{G}_m} \xhookrightarrow{\text{dom?}} \underline{H_d(\mathbb{A}^n)} \quad \text{INSANE}$$

$$= \left\{ \begin{array}{l} I \trianglelefteq k[x_1 \ldots x_n] \\ I \text{ homogeneous} \end{array} \right\} \quad \dim_k k[x_1 \ldots x_n]/I = d$$

$x_i \mapsto \underline{x_i - \alpha_i}$, $\alpha_i \in k$



ALG. TOP: $X \longrightarrow Y$

fibrant replacement! Path space!

BIAŁYNICKI-BIRULA DECOMPOSITIONS.

$$\mathbb{G}_m = \mathrm{Spec}\, k[t^{\pm 1}] \curvearrowright X \ (\text{proj.})$$

Def

$$X^+ = \mathrm{Map}(\mathbb{A}^1, X)^{\mathbb{G}_m} \qquad [\mathrm{Map}(I, X)]$$

BB decomposition

(functor!)

Thm (Drinfeld, J-Sivkiewicz 2017) X^+ is representable for all X locally finite type

$$\pi s = \mathrm{id}_{X^{\mathbb{G}_m}}$$

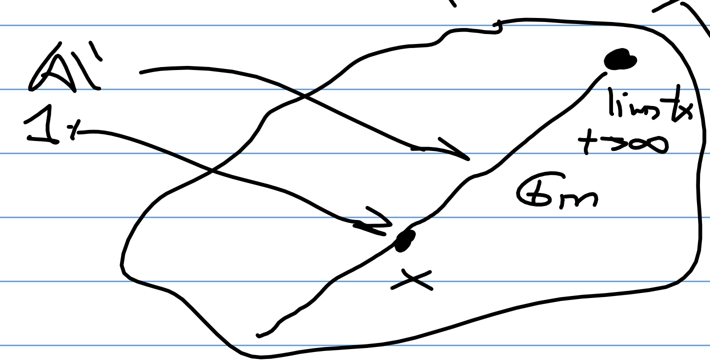
$$\begin{tikzcd} X^+ = \mathrm{Map}(\mathbb{A}^1, X)^{\mathbb{G}_m} \arrow[r, "i"] \arrow[d, "\pi"'] & X \\ X^{\mathbb{G}_m} \arrow[u, bend left, "s"] & \end{tikzcd}$$

FUNDAMENTAL DIAGRAM

On pts: $\mathbb{A}^1 \xrightarrow[\varphi]{\mathbb{G}_m\text{-eq}} X \longmapsto \varphi(1)$

$\lim_{t \to \infty} tx$

$\varphi(\infty)$



$$\mathbb{A}^1 = \mathbb{G}_m \cup \{\infty\}$$

$$x \in X^{\mathbb{G}_m} \rightsquigarrow \mathbb{A}^1 \to X$$

constant.

Obs: $X^+(k) \xrightarrow{\text{bij}} X(k)$

$\Uparrow$

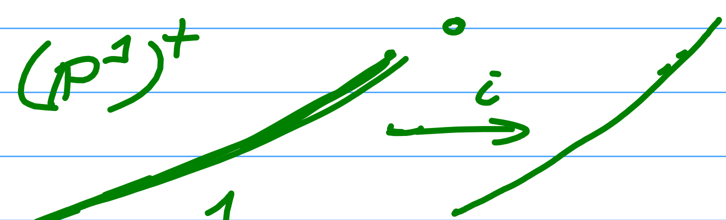
X proper

Thm (BB'73)

If X smooth complete

smooth $\begin{cases} X^+ = \bigsqcup_{i=1}^{r} X_i^+ & \text{connected components} \\ X^{\mathbb{G}_m} = \bigsqcup_{i=1} F_i & \text{connected components} \end{cases}$

$X^+ \xrightarrow[\lim]{} X^{\mathbb{G}_m}$ is an affine fibre bundle.



Ex $X = \mathbb{P}^1 \circlearrowleft \mathbb{G}_m$

$$\lim_{t\to\infty} tx = \begin{cases} \infty & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$X^+ = (\mathbb{P}^1 \setminus \{0\}) \sqcup \{0\}.$$

Thm (J): Σ sing. type

$S \subseteq \mathbb{P}^n \qquad (H, [S]) \in \Sigma$

We want to reduce S to dimension zero R, but so that $(H^+, [R]) \longrightarrow (H, [R])$ open immersion near $[R]$

KEY LEMMA BB decompositions:

$x \in X \circlearrowleft \mathbb{G}_m \qquad x \in X^{\mathbb{G}_m}$

$$\mathbb{G}_m \curvearrowright T_x X \rightsquigarrow (T_x X^+) = (T_x X)_{\geq 0}$$

Lemma: TFAE:

1. $T_x X^+ = T_x X \qquad (T_x X)_{<0} = 0$
2. $X^+ \xrightarrow{i} X$ is an open immersion near x.

$$\begin{tikzcd}
(H^+, [R]) \arrow[r, "i", "\text{open immersion}"'] \arrow[d, shift left, "\text{retraction}"] & (H, [R]) \\
(H^{\mathbb{G}_m}, [R]) \arrow[u, bend left] &
\end{tikzcd}$$

singularity type of $[R] \in H$ and $[R] \in H^{\mathbb{G}_m}$ are related up to retraction.

$$I_S = I \subseteq \Bbbk[x_1 \ldots x_n]$$

$$\downarrow$$

$$J := \underbrace{I \Bbbk[y_1 \ldots y_n]}_{(I)} + (y_1 \ldots y_n)^2 + (x_1 \ldots x_n)^{a+1} + \left(\sum_{i=1}^n x_i y_i\right) \qquad \Bbbk[x_1 \ldots x_n, y_1 \ldots y_n] \qquad a \gg 0$$

- sing type of I
- it lies in good cell

OPEN QUESTION: $T^1 = 0$

The above shows:

$\left(\sum_{i=1}^n x_i y_i\right) + (x_1 \ldots x_n)^{a+1}$ $\quad a \geq 1, \; n \geq 3$

is rigid as a proj. variety. WHY?

Ex: BB: X smooth proj

$|X^{\mathbb{G}_m}| < \infty$

$\Downarrow$

$\{p_1, \dots, p_r\}$

$X_i^+ \simeq \mathbb{A}^{n_i}$: $\Rightarrow X \supseteq \mathbb{A}^{\dim X}$

OPEN QUESTION $\mathbb{G}_a$:

X smooth proj.

$|X^{\mathbb{G}_a}| < \infty$

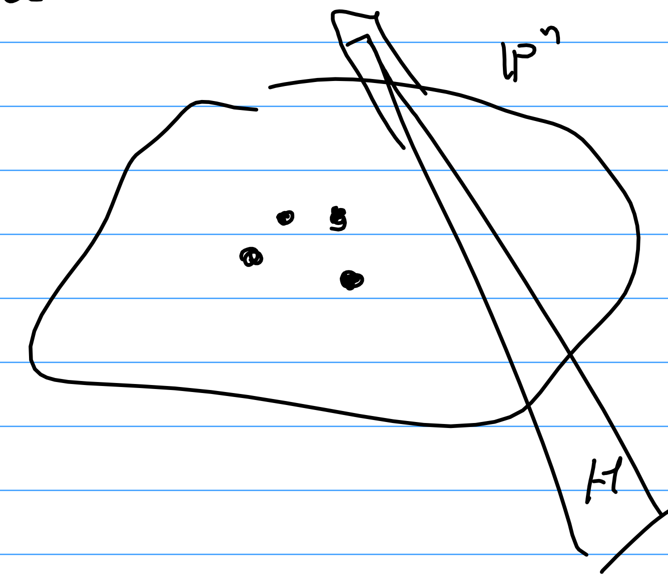
$\Downarrow$?

$X = \bigcup_{i=1} \mathbb{A}^{n_i} \Rightarrow X \supseteq \mathbb{A}^{\dim X}$.

BB decomposition for $\mathbb{G}_a$.

$X^+ = \mathrm{Map}(\mathbb{P}^1, X)^{\mathbb{G}_a}$?

$$\mathrm{Hilb}_d(\mathbb{P}^n) = \bigcup \mathrm{Hilb}_d(\mathbb{A}^n)$$



EXPLICIT EXAMPLES

NONREDUCED POINT (Ex 5.4)

$K = (x_1^2, x_1x_2, x_2^2(x_3+x_4), x_1x_4^3 + x_2(x_3+x_4)x_3^2)$

$I = K \cap (x_1, \dots, x_4)^4$

$J = I \cdot k[y_1, \dots, y_4] + (x_1, \dots, x_4)^5 + (y_1, \dots, y_4)^2 + (Q)$

$\uparrow$

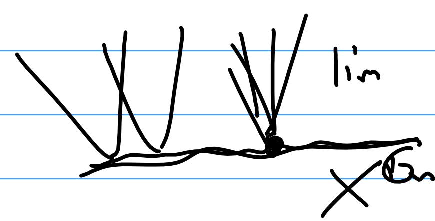
our pt

$$Q = \sum_{i=1}^{4} x_i y_i.$$

Path.

J.W. M Szachniewicz

$\mathrm{Hilb}_{13}(\mathbb{A}^6) \ni \operatorname{Spec} A$



A graded $\quad H_A = (1, 6, 6, 0, \dots)$

$A = B/\mathfrak{m}^3 \quad B$ graded Gorenstein $H_B = (1, 6, 6, 1, 0 \dots)$

COMPONENTS IN CHAR p

p odd $\quad p = 3, 5 \dots$

$I = (x_1y_1 + x_2y_2 + x_3y_3) + (y_1^p, y_2^p, y_3^p)$

$J = I + (x_1, x_2, x_3)^{p+1}.$

$[V(J)] \in \mathrm{Hilb}(\mathbb{A}^6_{\mathbb{F}_p})^{\mathbb{G}_m}$ rigid

Does not lift to $\mathbb{Z}/p^2$

Satisfies TNT $\rightarrow$

Does not lift to char $= 0$.