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Moduli spaces of quartic hyperelliptic K3 surfaces via K-stability

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Outline: (/ $\mathbb{C}$).

- Hyperelliptic K3 surfaces
- Two classical compactifications: GIT ① and Baily Borel ②
- K-stability: interpolates between ① & ②
- Compare to Laza - O'Grady.

§ Hyperelliptic K3 surfaces.

<u>Def</u>. A K3 surface is a normal proj. surface S with Du Val sing. s.t. $\omega_S \cong \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$.

canonical bundle.

Polarization: L ample line bundle over S.

Degree: $\deg(S, L) = (L^2) = 2g - 2$. $\longrightarrow$ genus of (S, L).

<u>Thm</u> (Mayer) Assume $|L|$ has no fixed component.

Then $|L|$ is base-pt-free, and $\phi_{|L|} : S \to \mathbb{P}^g$ finite.

<u>Case 1</u>. $\phi_{|L|}$ is an embedding. $\Longleftarrow$ $C \in |L|$ canonical curve.

<u>Case 2</u>. $\phi_{|L|}$ is 2-to-1. (hyperelliptic). $\Longleftarrow$ $C \in |L|$ hyp. ellip. curve.

$\mathrm{Im}(\phi_{|L|}) \subseteq \mathbb{P}^g$ is a rational surface of deg $(g-1)$.

Ex. • $\deg = 2$ $(g=2)$ $\quad \phi_{|L|}: S \xrightarrow{2:1} \mathbb{P}^2$. (general deg 2 polar. K3 = hyperelliptic)

Let $D \subseteq \mathbb{P}^2$ be the branch curve, then $K_S = \phi_{|L|}^*(K_{\mathbb{P}^2} + \frac{1}{2}D)$.

$K_S \sim 0 \Rightarrow -2K_{\mathbb{P}^2} \sim D \Rightarrow \deg D = 6$.

Hence S is a double cover of $\mathbb{P}^2$ branched along a sextic curve.

☆ • $\deg = 4$. $(g=3)$ (hyperelliptic)

Locally, $D = (f(x,y)=0) \subseteq \mathbb{C}^2$

$\Rightarrow S = (\underbrace{f(x,y)}_{\text{ADE.}} = \underbrace{z^2}_{\text{Du Val}}) \subseteq \mathbb{C}^3$.

$\phi_{|L|}: S \xrightarrow{2:1} T \subseteq \mathbb{P}^3$

$\deg T = 2 \Rightarrow T \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(1,1,2)$.

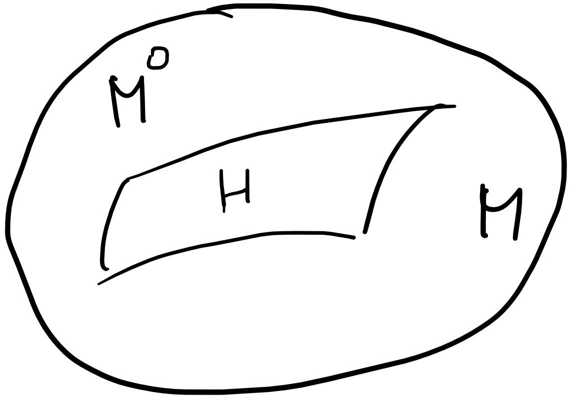
$D \sim -2K_T = \mathcal{O}_T(4) \Rightarrow D = (2,4)$-c.i. curve in $\mathbb{P}^3$.

D has at worst ADE sing.

§ Moduli spaces of quartic hyperelliptic K3s.

← quasi-proj. but not projective.

- M = coarse moduli space of quartic hyp. K3s.



$M^\circ \subseteq M$: open subset where $T \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$H = M \setminus M^\circ \subseteq M$: divisor where $T \cong \mathbb{P}(1,1,2)$.

- $S \xrightarrow{2:1} (T, D)$ determined.

M = moduli space of (T, D). $T \subseteq \mathbb{P}^3$ deg 2

$D \subseteq \mathbb{P}^3$ deg $(2,4)$ c.i.

M is birational to $(4,4)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$

$(2,4)$-c.i. in $\mathbb{P}^3$.

- Classical Compactifications

① GIT. $\mathbb{P}_{4,4} := \mathbb{P}(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(4,4)))$.

$\overline{M}^{GIT} := \mathbb{P}_{4,4} /\!/ \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \hookleftarrow M^\circ$. $\quad \mathrm{codim}_{\overline{M}^{GIT}} \overline{M}^{GIT} \setminus M^\circ \geq 2$.

But H is not contained in $\overline{M}^{GIT}$.

② Baily-Borel. (Hodge theory)

Global Torelli $\Rightarrow$ $\overset{\text{period map}}{p} : M \xrightarrow{\cong} \mathcal{D}/\Gamma$

ample Hodge line bundle

$\boxed{\dim M^* \setminus M \leq 1}$

$M^* = (\mathcal{D}/\Gamma)^* := \mathrm{Proj}\, R(\mathcal{D}/\Gamma, \lambda) \hookleftarrow M$.

<u>Q</u>. How to interpolate the birational map $\overline{M}^{GIT} \overset{bir}{\dashrightarrow} M^*$?

$$\begin{tikzcd} \overline{M}^{GIT} \arrow[rr, dashed, "bir"] & & M^* \\ & M^\circ \arrow[ul] \arrow[ur] & \end{tikzcd}$$

<u>A</u>. Laza-O'Grady VGIT (≅) ADL K-stability.

§ K-stability

K-stability $\Rightarrow$ K-polystability.
$\left(\text{K-ps} + |\mathrm{Aut}| < \infty \right)$. $\Downarrow$

- Algebro-geometric theory detects $\exists$ of KE metrics on (log) Fano
- Provides a "nice" moduli theory for (log) Fano.

In our case, consider $(\mathbb{P}^1 \times \mathbb{P}^1, cC)$, $C \in |\mathcal{O}(4,4)|$. branch curve

It is log Fano when $c \in (0, \frac{1}{2})$. coeff.

Lemma. If C has ADE sing., then $(\mathbb{P}^1 \times \mathbb{P}^1, cC)$ is K-stable for $\forall\, c \in (0, \frac{1}{2})$. (interpolation: $\mathbb{P}^1 \times \mathbb{P}^1$ K-polystable. $(\mathbb{P}^1 \times \mathbb{P}^1, \frac{1}{2}C)$ klt $\Rightarrow$ K-stable).

[Xu-Zhuang'19]: projective

[ADL'19]: construct proper good moduli space $\overline{M}_c^K \xhookleftarrow{\text{open}} M^0$.
Param. (X, cD) K-polystable.

Main results.

Thm 1 [APL]. If $c \in (0, \frac{1}{8})$, then $\overline{M}^K_c \cong \overline{M}^{GIT}$.

(All limiting surface $X \cong \mathbb{P}^1 \times \mathbb{P}^1$).

Thm 2 [APL]. $\exists\ 0 < c_1 < c_2 < \cdots < c_k < \frac{1}{2}$, s.t.

[ADL'19]

(1) $\overline{M}^K_c$ is indep. of choice of c if $c \in (c_i, c_{i+1})$.

(2) $\exists$ diagram

$$\begin{tikzcd} \overline{M}^K_{c_i-\varepsilon} \arrow[rr, dashed, leftrightarrow, "bir"] \arrow[dr] & & \overline{M}^K_{c_i+\varepsilon} \arrow[dl] \\ & \overline{M}^K_{c_i} & \end{tikzcd}$$

(étale locally VGIT).

(3) $\exists\ \overline{M}^K_{\frac{1}{2}-\varepsilon} \xrightarrow{bir} M^*$

this is the 'Looijenga $\mathbb{Q}$-Cartierization of $H^* \subseteq M^*$.

Thm 3 [ADL]. For $\forall c \in (0, \frac{1}{2})$, only surfaces X appearing in $\overline{M}^{K}_{c}$ are either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(1,1,2)$.

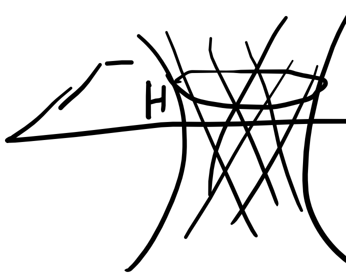
(Fails for $(3,3)$ - curves !)

H is a sm. $(1,1)$-curve

Ex $(c = \frac{1}{8})$. $\exists$ a special point $[4H] \in \overline{M}^{GIT}$

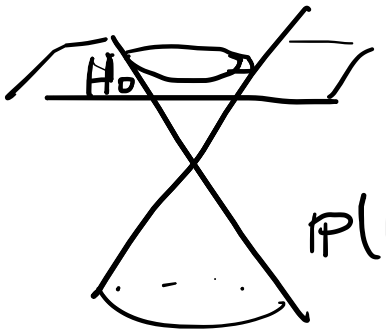
K-ps for $c = \frac{1}{8}$.

At $c = \frac{1}{8}$, we replace $(\mathbb{P}^1 \times \mathbb{P}^1, 4H) \rightsquigarrow (\mathbb{P}(1,1,2), 4H_0)$



$\mathbb{P}^1 \times \mathbb{P}^1$

$\rightsquigarrow$



$\mathbb{P}(1,1,2)$

At $c = \frac{1}{8} + \varepsilon$,
Replace $4H_0$ by deg 8 - curves D on $\mathbb{P}(1,1,2)$.

$H \overset{bir}{\dashleftarrow \dashrightarrow} D$

§ Compare to Laza-O'Grady (VGIT). $[T,D]$ $[T]$

L-O'G: $(\overset{\text{quadric}}{T}, \overset{(2,4)\text{-c.i.}}{D})$ parametrized by $\pi : \mathbb{P}(E) \to \mathbb{P}^9$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(2)))$

On $\mathbb{P}(E)$, $\exists$ two distinguished line bundle:

$\eta = \pi^* \mathcal{O}_{\mathbb{P}^9}(1)$, $\xi = \mathcal{O}_{\mathbb{P}(E)}(1)$.

$\eta + t\xi$ ample only when $t \in (0, \frac{1}{3})$.

$$\overline{M}(t) := \begin{cases} \mathbb{P}(E) /\!\!/_{\eta + t\xi} SL(4), & \text{if } t < \frac{1}{3} \\ \mathcal{P} /\!\!/_{N_t} SL(4), & \text{if } \frac{1}{3} \leq t \leq \frac{1}{2}. \end{cases}$$

$$\begin{tikzcd} & \mathcal{P} \arrow[dl] \arrow[dr] & \\ \mathbb{P}(E) \arrow[rr, dashed] & & \mathrm{Chow}_{(2,4)} \end{tikzcd}$$

Thm 4. [ADL]: $\exists\ \overline{M}_c^K \xrightarrow{\cong} \overline{M}(t)$ ← Laza-O'Grady VG 17

for $t = \dfrac{3c}{2c+2}$. $\forall\, c \in (0, \tfrac{1}{2})$

$\left(\forall t \in (0, \tfrac{1}{2})\right)$. $\mathrm{Chow}_{(2,4)} /\!/ SL(4)$.

 $/\!/$

Rmk: For $t = \frac{1}{2}$, $\overline{M}(\tfrac{1}{2}) \cong M^*$.

$$\text{Thm 4} \Rightarrow \overline{M}^K_{\frac{1}{2}-\varepsilon} \cong \overline{M}(\tfrac{1}{2} - \varepsilon') \xrightarrow[\text{wall}]{\text{VG 17}} \overline{M}(\tfrac{1}{2}) \cong M^*.$$

natural for $K \to \log CY$ wall crossing.

K-Walls : $\left\{\frac{1}{8}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{5}{16}, \frac{1}{3}, \frac{4}{11}\right\}$.

CM line bundles λ_c is linear in c.

(up to pos. multiple).

Interesting future work / direction:

① $(\mathbb{P}^3, cS)$ ← S deg 4. ← $\mathbb{P}^3$ ↕ LOG

② $(3,3)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$.

$K \neq$ VGIT in general.

related $g=4$ Hassett-Keel

(not the same).