The Chow Rings of $\mathcal{M}_7$, $\mathcal{M}_8$, and $\mathcal{M}_9$

joint work with

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<u>Conventions</u>: Work over an algebraically closed field of characteristic 0 or $p > 5$.

Chow rings are taken with $\mathbb{Q}$-coefficients.
$$A^*(X) = A^*(X;\mathbb{Q}).$$

<u>The tautological ring:</u>

$$\begin{tikzcd} \mathcal{C} \arrow[d, "f"] \\ \mathcal{M}_g \end{tikzcd}$$
universal curve

<u>Defn</u>: The tautological ring $R^*(\mathcal{M}_g) \subseteq A^*(\mathcal{M}_g)$ is the $\mathbb{Q}$-subalgebra generated by the kappa classes
$$\kappa_i = f_*(c_1(\omega_f)^{i+1}).$$

1. The tautological ring is known for $g \leq 23$ by Faber, Faber–Zagier.
2. "geometric" classes are often tautological.
   <u>e.g.</u> $[\mathcal{H}_g] \in R^{g-2}(\mathcal{M}_g)$.

Question: Is $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$?

<u>Theorem</u>: 1. ($g=2$) (Mumford, 1983) yes, $A^*(\mathcal{M}_2) = \mathbb{Q}$

2. ($g=3,4$) (Faber, 1990) yes, $A^*(\mathcal{M}_3) = \mathbb{Q}[\kappa_1]/(\kappa_1^2)$
$A^*(\mathcal{M}_4) = \mathbb{Q}[\kappa_1]/(\kappa_1^3)$

3. ($g=5$) (Izadi, 1995) yes, $A^*(\mathcal{M}_5) = \mathbb{Q}[\kappa_1]/(\kappa_1^4)$.

④ $(g=6)$ (Penev-Vakil, 2015)

$$A^*(\mathcal{M}_6) \cong \mathbb{Q}[\kappa_1,\kappa_2]\big/\left(127\kappa_1^3-2304\kappa_1\kappa_2,\ 113\kappa_1^4-36864\kappa_2^2\right)$$

<u>Theorem</u> (Van Zelm 2018): The fundamental class of the bielliptic locus in genus 12 $[\mathcal{B}_{12}]\in A^*(\mathcal{M}_{12})$ is not tautological! $C \xrightarrow{2:1} E$

What happens in the gap, $7\le g\le 11$?

<u>Theorem (C.-H. Larson):</u>

For $7\le g\le 9$, $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$

$$A^*(\mathcal{M}_7)=\mathbb{Q}[\kappa_1,\kappa_2]/I_7,\quad I_7 \overset{\text{gen}}{=} \begin{cases} 2423\kappa_1^2\kappa_2 - 52632\kappa_2^2 \\ 1152000\kappa_2^2 - 2423\kappa_1^4 \\ 16000\kappa_1^3\kappa_2 - 731\kappa_1^4. \end{cases}$$

$$A^*(\mathcal{M}_8)=\mathbb{Q}[\kappa_1,\kappa_2]/I_8,\quad I_8 \overset{\text{gen}}{=} \begin{cases} 714894336\kappa_2^2 + 55211328\kappa_1^2\kappa_2 - 1058587\kappa_1^4 \\ 62208000\kappa_1\kappa_2^2 - 95287\kappa_1^5 \\ 144000\kappa_1^3\kappa_2 - 5617\kappa_1^5. \end{cases}$$

$$A^*(\mathcal{M}_9)=\mathbb{Q}[\kappa_1,\kappa_2,\kappa_3]/I_9,\quad I_9 \overset{\text{gen}}{=} \begin{cases} 5195\kappa_1^4 + 3644694\kappa_1\kappa_3 + 749412\kappa_2^2 - 265788\kappa_1^2\kappa_2 \\ 33859814400\kappa_2\kappa_3 - 95311440\kappa_1^3\kappa_2 + 2288539\kappa_1^5 \\ 19151377\kappa_1^5 + 16929907200\kappa_1\kappa_2^2 - 114345520\kappa_1^3\kappa_2 \\ 1422489600\kappa_3^2 - 983\kappa_1^6 \\ 1185408000\kappa_2^3 - 47543\kappa_1^6. \end{cases}$$

Theorem (C.-H. Larson) The bielliptic locus $[\mathcal{B}_{10}] \in A^*(\mathcal{M}_{10})$ is tautological.

Strategy: Stratify $\mathcal{M}_g$ by gonality.

$$\mathcal{M}_g^k := \{ [C] \in \mathcal{M}_g : \exists\, \alpha : C \to \mathbb{P}^1 \text{ of degree} \leq k \}$$

$g=7$, $g=8$: $\mathcal{M}_g^2 \subseteq \mathcal{M}_g^3 \subseteq \mathcal{M}_g^4 \subseteq \mathcal{M}_g^5 = \mathcal{M}_g$

$g=9$: $\mathcal{M}_9^2 \subseteq \mathcal{M}_9^3 \subseteq \mathcal{M}_9^4 \subseteq \mathcal{M}_9^5 \subseteq \mathcal{M}_9^6 = \mathcal{M}_9$

Show classes supported on $\mathcal{M}_g^k$ are tautological modulo classes supported on $\mathcal{M}_g^{k-1}$.

Fact: (Faber) $[\mathcal{M}_g^k] \in A^*(\mathcal{M}_g)$ is tautological.

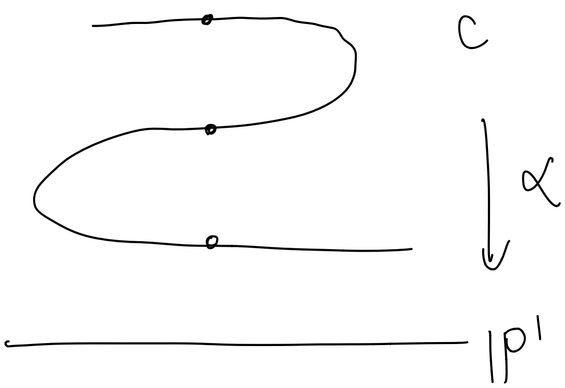
Inductive process starting from the bottom.

Hyperelliptic curves: $\mathcal{M}_g^2$

$2g+2$ points on $\mathbb{P}^1$. Fix 3 of them by an elt of PGL_2.

$$\mathbb{A}^{2g-1} \xrightarrow{\text{finite}} \underbrace{\mathbb{A}^{2g-1}/S_{2g+2}} \implies A^*(\mathcal{M}_g^2) = \mathbb{Q}.$$

Trigonal Curves: $\mathcal{M}_g^3 \setminus \mathcal{M}_g^2$



Use the Hurwitz space moduli space $\mathcal{H}_{3,g}$ of degree 3 covers of $\mathbb{P}^1$ by smooth genus g curves.

$g \geq 5$, $\mathcal{H}_{3,g} \cong \mathcal{M}_g^3 \setminus \mathcal{M}_g^2$.

Thm: (Patel-Vakil): $A^*(\mathcal{H}_{3,g})$ is generated by κ_1 for $g \geq 4$.

Thm: (C.-H. Larson) $A^*(\mathcal{H}_{3,g}) = \begin{cases} \mathbb{Q}[\kappa_1]/(\kappa_1^2) & g = 4,5 \\ \mathbb{Q}[\kappa_1]/(\kappa_1^3) & g \geq 6 \end{cases}$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathbb{F}_n = \mathbb{P}E \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd} \qquad H^0(\mathcal{O}_{\mathbb{P}E}(3) \otimes \det E^\vee) \Big/ \operatorname{Aut} E \times \operatorname{Aut} \mathbb{P}^1$$

- Producing relations is "key" in higher gonality.
- So far, we have shown all classes supported on $\mathcal{M}_g^3$ are tautological.

$\mathcal{M}_g^4$, $\mathcal{M}_g^5$, and the Hurwitz space

Again, the Hurwitz space

- $\mathcal{H}_{k,g}$ moduli space of smooth genus g degree k covers of $\mathbb{P}^1$.

$k = 4, 5$.

Forgetful map: $\beta: \mathcal{H}_{k,g} \to \mathcal{M}_g$

induced map $\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1}) \to \mathcal{M}_g \setminus \mathcal{M}_g^{k-1}$
is proper.

$$A_*(\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1})) \xrightarrow{\text{surjective}} A_*(\mathcal{M}_g^{k} \setminus \mathcal{M}_g^{k-1})$$

<u>Goal</u>: Understand $A_*(\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1}))$.

- What are the generators?
- Do the generators push forward to tautological classes?

Can no longer expect things to work out so cleanly.

1. Bielliptic curves are tetragonal. Van Zelm's result says we must expect genus dependence.

2. $\mathrm{Pic}(\mathcal{H}_{k,g}) \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus 2}$, for $k = 4, 5$. (Deopurkar–Patel)
   $\Rightarrow$ not all classes on $\mathcal{H}_{k,g}$ are pullbacks of taut classes on $\mathcal{M}_g$.

<u>Intersection Theory on $\mathcal{H}_{k,g}$:</u>

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathcal{P} \arrow[dl, "\pi"] \\ & \mathcal{H}_{\kappa,g} & \end{tikzcd}$$

univ. curve f π univ $\mathbb{P}^1$-bundle

<u>Def^n:</u> The tautological ring $R^*(\mathcal{H}_{\kappa,g}) \subseteq A^*(\mathcal{H}_{\kappa,g})$ is the $\mathbb{Q}$-subalgebra generated by

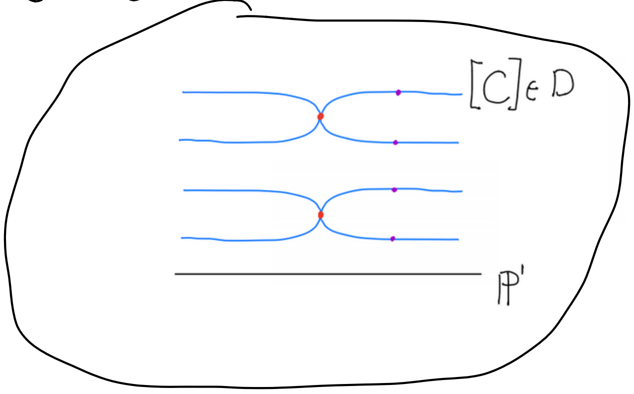
$$f_*\left(c_1(\omega_f)^i \cdot \alpha^* c_1(\omega_\pi)^j\right).$$

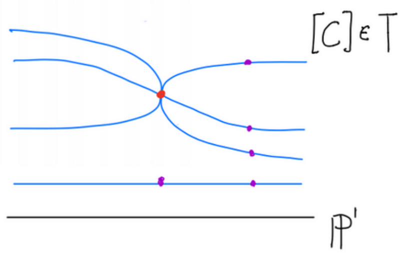
<u>Example:</u> $R^1(\mathcal{H}_{\kappa,g})$ is generated by

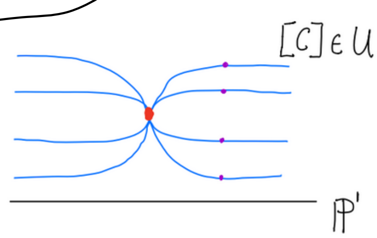
$$\kappa_1 = f_*\left(c_1(\omega_f)^2\right), \quad e_1 = f_*\left(c_1(\omega_f)\cdot \alpha^* c_1(\omega_\pi)\right)$$

and $\quad f_*\left(\alpha^* c_1(\omega_\pi)^2\right) = 0.$

<u>Ramification type classes:</u> Strata parametrizing covers that have certain ramification profiles.







Deopurkar-Patel: For $k=4,5$, $A^1(\mathcal{H}_{k,g}) = R^1(\mathcal{H}_{k,g}) = \mathbb{Q}\langle T, D\rangle$.

Theorem: (Faber-Pandharipande) The pushforwards of ramification type cycles to $\mathcal{M}_g$ are tautological.

Theorem: (C.-H. Larson) For $k=4,5$, pushforwards of tautological classes are tautological:

$$R_*(\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}^{k-1}_g)) \longrightarrow R_*(\mathcal{M}_g \setminus \mathcal{M}^{k-1}_g)$$

Pf sketch: ① Write down a nice generating set for $R^*(\mathcal{H}_{k,g})$.

② Find all of the relations in codim up to $\sim g/k$.

③ $R^*(\mathcal{H}_{k,g})$ is generated as a module over $\mathbb{Q}[\kappa_1, \kappa_2]$ by ramification type classes.

Consequence: We've now fully transformed the problem on $\mathcal{M}_g$ to a problem on $\mathcal{H}_{k,g}$.

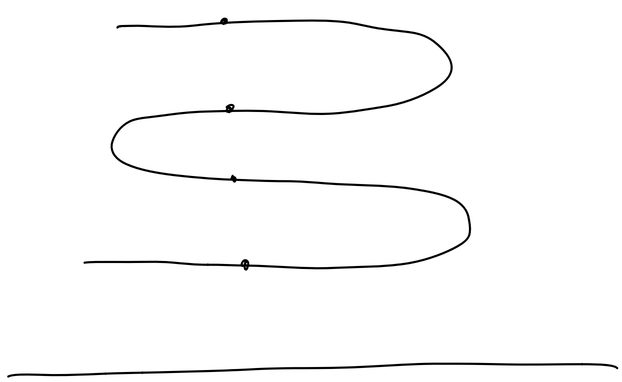
New Goal: Show that

$$A^*(\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1})) = R^*(\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1}))$$

(extra work for $g=9$, $k=4$)

On $\mathcal{H}_{4,g}$

Let $\alpha : C \to \mathbb{P}^1$ be a 4:1 cover.



$$\begin{array}{c} C \\ \downarrow \alpha \\ \mathbb{P}^1 \end{array}$$

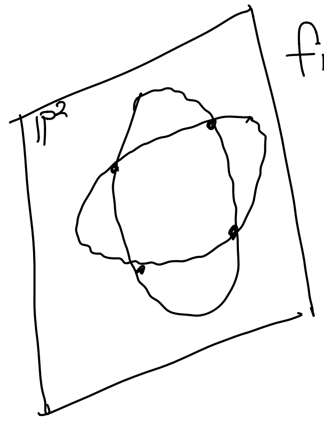
- Geometric Riemann-Roch says the fibers span a $\mathbb{P}^2$ in the canonical embedding.

$\Rightarrow$ The fibers sweep out a $\mathbb{P}^2$-bundle

Algebraically: $0 \to \mathcal{O}_{\mathbb{P}^1} \to \alpha_* \mathcal{O}_C \to E_\alpha^\vee \to 0$

E_α
- rank 3
- degree $g+3$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\alpha"] & \mathbb{P}E_\alpha^\vee \arrow[r] \arrow[dl, "\gamma"] & \mathbb{P}^{g-1} \\ \mathbb{P}^1 & & \end{tikzcd}$$



fiber of $\mathbb{P}E_\alpha^\vee \xrightarrow{\gamma} \mathbb{P}^1$

- 4 pts as base locus of pencil of conics.

- C is the zero locus of a section
$$\eta \in H^0(\mathbb{P}E_\alpha^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(2) \otimes \gamma^* F_\alpha^\vee)$$
$$= H^0(\mathbb{P}^1, \mathrm{Sym}^2 E \otimes F_\alpha^\vee).$$

- F_α is a rank 2 and deg $F_\alpha = g+3$.

To a cover $\alpha: C \to \mathbb{P}^1$, associate two vector bundles, E_α, F_α on $\mathbb{P}^1$.

$$E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3) \qquad \sum e_i = g+3.$$

$$F = \mathcal{O}(f_1) \oplus \mathcal{O}(f_2) \qquad \sum f_i = g+3.$$

$\mathcal{H}_{4,g}$ is stratified by the splitting types of the pairs E_α, F_α.

Example: $g = 7$, $\deg E = \deg F = 10$

	Splitting types $(e_1, e_2, e_3), (f_1, f_2)$	Expected codim $h^1(\operatorname{End} E) + h^1(\operatorname{End} F)$	Codim in $\mathcal{H}_{4,7}$ $=$ exp. codim $- h^1(F^\vee \otimes \operatorname{Sym}^2 E)$
good	$(3,3,4)$ $(5,5)$	0	0
good	$(3,3,4)$ $(4,6)$	1	1
bad	$(2,4,4)$ $(4,6)$	3	2
bad	$(2,3,5)$ $(4,6)$	4	3

ugly (1,4,5) (2,8) 10 2

B = moduli of pairs (E,F)

$[\alpha : C \to \mathbb{P}^1] \mapsto (E_\alpha, F_\alpha)$.

$\mathcal{H}_{4,7} \longrightarrow B$

3 step process:

① (Blue part) $\mathcal{H}^\circ = \{\alpha : h^1(F_\alpha^\vee \otimes \mathrm{Sym}^2 E_\alpha) = 0\}$

B° = moduli of (E,F) with $h^1(F^\vee \otimes \mathrm{Sym}^2 E) = 0$.

$\mathcal{H}^\circ \to B^\circ$ vector bundle morphism.

Chow rings of vbs are easy.

H. Larson computed Chow ring of B

(2) (Gray part) • Get rid of curves of lower gonality. All the gray curves are hyperelliptic: $C \xrightarrow{2:1} \mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^1$

(3) (Red part) Two stratum

$$\Sigma_1 = (2,4,4) \quad (4,6)$$

$$\Sigma_2 = (2,3,5) \quad (4,6)$$

(a) Show $A^*(\Sigma_i)$ is generated by taut classes

$$\Sigma_i \underset{\text{open}}{\subseteq} \mathbb{A}^N / G_i$$

(b) $[\Sigma_i] \in A^*(\mathcal{H}_{4,7} \setminus \beta^{-1}(\mathcal{M}_7^3))$.

(i) Σ_1 is the only statum w/

$E = (2,4,4)$

$\Sigma_2 \quad E = (2,3,5)$

(ii) $\operatorname{codim} \Sigma_1 = 2 = h^1(\operatorname{End} E)$

$\operatorname{Codim} \Sigma_2 = 3 = h^1(\operatorname{End} E)$

$\Rightarrow$ H. Larson's splitting loci formula imply $[\Sigma_i]$ are tautological.

<u>Example:</u> ($g=12$, bielliptic)

$E = (2,6,7)$ $\qquad F = (4,11)$

Bielliptics codim 4 in $\mathcal{H}_{4,12}$. But neither $h^1(\operatorname{End} E)$ nor $h^1(\operatorname{End} F)$ is 4, so (ii) above fails.

~15 mins.

Similar ideas work for $k=5$.

Hexagonal genus 9 curves:

Theorem (Mukai): $\mathcal{M}_9 \setminus \mathcal{M}_9^5$ is the quotient $\left[\dfrac{G(9,14) \setminus \Delta}{PGSp_6}\right]$.

Using equivariant intersection theory, we show that the Chow ring of this quotient is generated by tautological classes. The argument is very similar to Penev-Vakil's argument when $g=6$.