Logarithmic Resolution of Singularities

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A joint project with D. Abramovich and J. Włodarczyk on resolution of singularities of morphisms and log varieties.

References:

- [ATW17] "Principalization of ideals on logarithmic orbifolds", JEMS **22**, 2020.
- [ATW20] "Relative desingularization and principalization of ideals".
- [ATW19] "Functorial embedded resolution via weighted blowings up".

Classical resolution

- For simplicity, we only consider varieties over a field k. The characteristic is zero. Also, can take $k = \mathbb{C}$ and work with analytic spaces (using the usual topology instead of the étale one).
- Resolution of singularities associates to an integral variety Z a modification (i.e. proper birational) $Z_{\mathrm{res}} \to Z$ with a smooth Z_{res}.
- Hironaka 1964 (the Fields medal work): a resolution exists.
- Hironaka, Giraud 70ies: simplifications, maximal contact.
- Villamayor, Bierstone-Milman 80ies-90ies: algorithmic and canonical resolution.
- Włodarczyk 2005: smooth-functoriality, i.e. $Z'_{\mathrm{res}} = Z' \times_Z Z_{\mathrm{res}}$ for any smooth $Z' \to Z$. This both simplifies the arguments and has stronger applications (e.g. equivariant resolution).

Relative and logarithmic resolution

- [ATW17] The classical algorithm has a logarithmic analogue associating to each generically log smooth log variety X a modification $X_{\mathrm{res}} \to X$ with a log smooth log DM stack X_{res}. It is functorial w.r.t. log smooth morphisms $Y \to X$.
- [ATW20] The same logarithmic resolution algorithm applies to a morphism $f\colon X \to B$ of log schemes constructs $X_{\mathrm{res}} \to X$ with a log smooth $X_{\mathrm{res}} \to B$, but it can fail when $\dim(B) > 1$.
- The new ingredient: there exists a modification $h\colon B' \to B$ s.t. the algorithm does not fail for the base change $f'\colon X' \to B'$. Moreover, $X'_{\mathrm{res}} \to X_{\mathrm{res}}$ is compatible with further base changes $B'' \to B'$.
- In the current version h is not canonical, so resolution of morphisms is only relatively functorial.
- Work in progress: h can be chosen canonically.

Plan

1. Classical resolution
   - General framework
   - Induction on dimension
2. Logarithmic geometry
3. Logarithmic algorithms

Embedded resolution

- All canonical methods before [ATW17] construct essentially the same algorithm built on Hironaka's framework. Everything is done locally and glues due to the functoriality.
- The resolution is embedded: one (locally) embeds X into a manifold (i.e. a smooth variety) M. To the pair (M, X) one associates a modification of manifolds $f : M_{\mathrm{res}} \to M$ and $X_{\mathrm{res}} \hookrightarrow X \times_M M_{\mathrm{res}}$ is a certain transform of X under f.
- Functorial embedded resolution implies functorial non-embedded one because an embedding $X \hookrightarrow M$ with minimal $\dim(M)$ is unique (étale) locally.

Main choices

The following choices are done in the classical resolution:

(1) Class of modifications: the algorithm iteratively blows up submanifolds $V \subset M$. Notation: $f_i \colon M_{i+1} = \mathrm{Bl}_{V_i}(M_i) \to M_i$.

(2) Transforms: one pullbacks X and subtracts a multiple of the exceptional divisor: $X_{i+1} = f_i^{-1}(X_i) - dE_{f_i}$.

(3) Choice of centers: the order $d = d_1$ of $I = I_X$ at $x \in M$ is a (very crude) primary invariant.

(4) The history: to avoid loops the algorithm encodes history in the iterated exceptional sncd E. The number $s(x)$ of its components at x is another primary invariant.

(5) Induction: one iteratively restricts to hypersurfaces of maximal contact, getting induction on $n = \dim(M)$. The actual invariant, whose maximal locus is blown up, is closer to $(d_1, s_1, d_2, s_2, \dots, d_n)$ with the lex order.

History and a dream algorithm

The classical algorithm has a subtle inductive structure and encodes history of the process in the boundary. With our choices a no-history algorithm does not exist:

Example (No progress.)

Let $\phi = x^2 - yzt$ and $X = V(\phi)$ in $M = \mathbb{A}^4$. Then $V = 0$ is the only smooth S_3-equivariant subscheme containing 0 in X_{sing}, but $M' = \mathrm{Bl}_V(M)$ has charts with $X' = f^{-1}(X) - 2E$ having the same singularity, e.g. in M'_y we have $\phi = (x'y')^2 - y'(y'z')(y't') = y'^2(x'^2 - y'z't')$.

A similar computation shows that blowing up the pinch point of Whitney umbrella $V(x^2 - y^2z)$ yields a pinch point again.

Using weighted blow ups allows to [ATW19] to construct a dream algorithm which just iteratively blows up the maximal invariant locus, so that the invariant drops. No history is needed there.

The boundary

- After a blow up $f\colon M' \to M$ each point $x \in E = V(t)$ has a god given coordinate t (unique up to a unit) coming from the history of the resolution. One only uses coordinate systems which include t.
- Inductively, for a sequence $f_i\colon M_{i+1} \to M_i$ we set $E_{i+1} = f_i^{-1}(E_i) \cup E_{f_i}$ and call it the accumulated boundary of M.
- We always work with coordinates $t_1, \ldots, t_n$ s.t. $V_i = V(t_{i_1}, \ldots, t_{i_l})$ and $E_i = (t_{n-r+1} \ldots t_n)$. So, E_i is an snc (simple normal crossings) divisor and V_i has simple normal crossings with E_i (lies in few components and is transversal to others).
- We call the boundary coordinates exceptional or monomial and denote them $m_1, \ldots, m_r$. So, $(t_1, \ldots, t_n) = (t_1, \ldots, t_{n-r}, m_1, \ldots, m_r)$.

The role of the boundary

Good news:

- Using canonical monomial coordinates decreases choices, makes the construction more canonical, helps to avoid loops.
- Boundary can accumulate parts of $I = I_X$: we set $I = I^{\mathrm{mon}} I^{\mathrm{pure}}$, where $I^{\mathrm{mon}} = (m_1^{l_1} \dots m_r^{l_r})$ and I^{pure} is purely non-monomial.

Bad news/another side of the same coin:

- Must treat E and monomial coordinates with a special care.
- Less possibilities for coordinates, centers must have snc with E.

Remark

Many technical complications of the classical algorithm are due to a bad separation of regular and exceptional coordinates because both are used to define the order.

Principalization

- All algorithms work algebraically with $I = I_X$ and solve the following principalization problem: find a sequence of submanifold blow ups $(M_n, E_n) \to \cdots \to (M, E)$ such that $I_n = I_X \mathcal{O}_{X_n}$ is invertible and monomial (i.e. supported on E_n).
- Magic: the last non-empty strict transform $X_l \subset M_l$ of X equals to V_l. So, it is smooth and transversal to E_l.
- Thus, principalization implies resolution $X_l \to X$ and even resolves the boundary $E_l|_{X_l}$ (a strong smell of a log geometry).
- A great profit: working with ideals provides a lot of flexibility.

Order reduction

- The main invariant of the algorithm is $d = \mathrm{ord}(I^{\mathrm{pure}})$, where $\mathrm{ord}(J) = \min_{f\in J}\mathrm{ord}(f)$. For example, $\mathrm{ord}(x^2 - yz^2)$ is 2 at any point of the z-axis and $\mathrm{ord}_O(x^5 + y^7, x^3z^3) = 5$.
- One works with marked (or weighted) ideals (I, d) where $d \geq 1$, only uses $M' = Bl_V(M)$ with $V \subseteq (I, d)_{\mathrm{sing}} := \{x \in M | \mathrm{ord}_x(I) \geq d\}$, and updates I by $I' = (I\mathcal{O}_{M'})I_{E'}^{-d}$. E.g., as we have computed earlier $(x^2 - yzt, 2)' = (x'^2 - y'z't', 2)$ on the y-chart.
- Order reduction finds a sequence $M_n \to \cdots \to M$ of such (I, d)-admissible blow ups so that $(I_n, d)_{\mathrm{sing}} = \emptyset$. Its existence implies principalization just by taking $d = 1$.

Remark

The so-called max order case when $d = \mathrm{ord}(I^{\mathrm{pure}})$ is the main one. It implies the general one relatively easily (and characteristic free). One has to consider the general case due to a bad (inductive) karma.

Maximal contact

- The miracle enabling induction on dimension is that in the maximal order case, order reduction of (I, d) is equivalent to that of $(C(I)|_H, d!)$, i.e. a blow up sequence reduces the order of (I, d) iff it reduces the order of $(C(I)|_H, d!)$. Here $C(I)$ is a coefficient ideal and H is a hypersurface of maximal contact.
- The Main Example: if $I = (t^d + a_2 t^{d-2} + \cdots + a_d)$ with $t = t_1$ and $a_i(t_2, \ldots, t_n)$, then $H = V(t)$ and $C(I) = (a_2^{d!/2}, \ldots, a_d^{d!/d})$.

Remark

(i) Why coefficient ideal? Because, unlike $C(I)|_H$, the stupid restriction $I|_H = (a_d)|_H$ looses a lot of information.
(ii) Each coefficient a_i has natural weight i.
(iii) No problem to have $a_1 = 0$ in characteristic zero (enough $d \in k^\times$).

Derivations

The main tool for a choice-free description of the algorithm is the derivation ideals $D(I) = D^1(I)$ generated by the elements of I and their derivations, and its iterations $D^n(I) = D(D^{n-1}(I))$. Note that $\mathrm{ord}_x(I) = \mathrm{ord}_x(D(I)) + 1$ for $x \in V(I)$. The derivation provides a conceptual way to define all basic ingredients excluding the monomial ones:

(1) $\mathrm{ord}_x(I)$ is the minimal d such that $D^d(I_x) = \mathcal{O}_x$.

(2) Maximal contact is any $H = V(t)$, where t is a regular coordinate in $D^{d-1}(I_x)$ (in particular, H is smooth).

(3) The coefficient ideal $C(I)$ is just $\sum_{i=0}^{d-1}(D^i(I))^{d!/(d-i)}$.

Remark

The only serious difficulty in proving canonicity of the algorithm is to show independence of the choice of a maximal contact.

Log derivations

The module of logarithmic derivations $D^{\log}$ is spanned by $m_j\partial_{m_j}$ and ∂_{t_i} for regular t_i's. These are the derivations preserving E (i.e. taking I_E to itself). For almost all needs it is easier and more conceptual to use $D^{\log}$, but it does not compute the order. This is why one has to use the usual derivations and runs into two complications as follows.

Choice of the maximal contact

(1) If E is not transversal to H then $E|_H$ makes no sense for us, hence we loose the control on the choice of centers having snc with E.

Solution: new boundary is transversal to H (and any center lying in it), so first iteratively reduce the order of I along the locus where the multiplicity s of the old boundary is maximal (practically, work with $I + I^d_{E(s)}$). Thus, our primary invariant is (d, s^{old}) ordered lexicographically.

Monomial contribution to the order

(2) When $\mathrm{ord}(I) \geq d$ but $\mathrm{ord}(I^{\mathrm{pure}}) < d$ we cannot proceed by looking only at I^{pure}. This happens because I^{mon} contributes to the order and causes that $(I, d)_{\mathrm{sing}} \neq \emptyset = (I^{\mathrm{pure}}, d)_{\mathrm{sing}}$.

Solution:
1. Reduce $e = \mathrm{ord}(I^{\mathrm{pure}})$ only along the locus where $\mathrm{ord}(I^{\mathrm{mon}}) \geq d - e$. Practically, we resolve the so-called companion ideal, which is the weighted sum of (I^{pure}, e) and $(I^{\mathrm{mon}}, d - e)$.
2. Once $e = 0$ (i.e. $I^{\mathrm{pure}} = (1)$), apply a purely combinatorial step to I^{mon}.

What is the boundary?

To proceed let us try to understand what the boundary really is.

- Unlike the embedded scheme X, I think it is wrong to view E as a subscheme of M (though it is determined by it). This is hinted at by functoriality: we consider blow ups $(M', E') \to (M, E)$ which do not take E' to E: one has that $f^{-1}(E) \hookrightarrow E'$ instead of $E' \hookrightarrow f^{-1}(E)$.
- The boundary is also determined by the sheaf of monomials $\mathcal{M}_M = \mathcal{M}_M(\log\ E) = \mathcal{O}^\times_{M\setminus E} \cap \mathcal{O}_M \subset (\mathcal{O}_M, \cdot)$ consisting of elements invertible outside of E. This gives the right functoriality: $f^*(\mathcal{M}_M(\log\ E)) \to \mathcal{M}_{M'}(\log\ E')$.
- In fact, the sheaf of monomials $\mathcal{M}_M(\log\ E)$ is precisely what we need from E!
- Locally $\mathcal{M}_M = \mathcal{O}^\times_M \times \mathbb{N}^s$ but this splitting (called a monoidal chart) is non-canonical: it is given by fixing exceptional coordinates $m_1, \dots, m_s$.

Logarithmic varieties

Definition

A logarithmic variety $(X, \mathcal{M}_X)$ consists of a variety X with a sheaf of monoids $\mathcal{M}_X$ and a homomorphism $\alpha_X\colon \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$ such that $\mathcal{M}_X^\times = \alpha_X^{-1}(\mathcal{O}_X^\times)$. A morphism is a compatible pair $f\colon X' \to X$ and $f^*\mathcal{M}_X \to \mathcal{M}_{X'}$.

- The example covering our needs is $(X, \mathcal{M}_X(\log D))$ for a divisor D. Morphisms are $f\colon X' \to X$ s.t. $f^{-1}(D) \hookrightarrow D'$.
- Many constructions extend to log geometry, e.g. $\Omega_{(X,M_X)}$ is generated by Ω_X and elements δm for $m \in M_X$ subject to relations $d\alpha(m) = \alpha(m)\delta m$ (i.e. δm is the log differential of m).
- One also defines log smooth morphisms. As in the classical case, they have locally free sheaves of relative differentials of expected rank.

Toroidal varieties

- Log smooth varieties are just toroidal ones: étale (analytically or formally) locally it suffices to work with the chart $X = \mathrm{Spec}\ \mathbb{C}[M][t_1, \dots, t_l]$ at its origin O, where M is the monoid of integral points in a rational polyhedral cone. The log structure is induced by M, and $\Omega_{(X,M)}$ is freely generated by dt_i and δm_i, where $\{m_i\}$ is any basis of M^{gp}. The classical notation is (X, U) or (X, D) with $D = \cup_{m \in M} V(m)$ and $U = X \setminus D$.
- In other words, $\mathcal{O}_{X,x} = \mathbb{C}[[M]][[t_1, \dots, t_l]]$. We view t_i as regular coordinates and all elements of M as monomial coordinates.
- Monomial democracy: M does not have to be free anymore and there is no canonical base of M^{gp}.

Toroidal morphisms

Log smooth morphisms of toroidal varieties are just toroidal morphisms, i.e. they are (étale-locally) modelled on toric maps and formally-locally look as

$$\mathbb{C}[[M]][[t_1,\dots,t_r]] \hookrightarrow \mathbb{C}[[N]][[t_1,\dots,t_n]], M \hookrightarrow N.$$

Example

(i) Semistable maps with appropriate log structures. For example, $\mathrm{Spec}\ \mathbb{C}[x,y] \to \mathrm{Spec}\ \mathbb{C}[\pi]$ given by $\pi = x^a y^b$ is log smooth for the log structures given by $x^{\mathbb{N}} \times y^{\mathbb{N}}$ and $\pi^{\mathbb{N}}$. The relative differentials are spanned by $\delta x = -\frac{b}{a}\delta y$.
(ii) Kummer log-étale covers are obtained when $N \subset \frac{1}{d}M$ and $r = n$. Relative differentials vanish. Finite but usually non-flat, e.g. $\mathrm{Spec}\ \mathbb{C}[x,y] \to \mathrm{Spec}\ \mathbb{C}[x^2, xy, y^2]$ with the log structures of monomials in x, y.

Some remarks

Remark

Toroidal morphisms are log smooth maps of log smooth varieties. In a sense, log geometry extends both to the non-smooth case (and $\mathbb{Z}$-schemes).

Remark

The most interesting feature of the new algorithm is functoriality w.r.t. Kummer log-étale covers, e.g. obtained by extracting roots of the monomial coordinates in the classical setting, or obtained by extracting roots of π is in the semistable reduction case. This is out of reach (and unnatural) for the classical algorithms.

Main results

Ignoring an orbifold aspect, our main result is:

Theorem (Log principalization)

Given a toroidal variety X with an ideal $I \subset \mathcal{O}_X$ there exists a sequence of admissible blowings up of toroidal varieties $X_n \to \cdots \to X$ such that the ideal $I\mathcal{O}_{X_n}$ is monomial. This sequence is compatible with log smooth morphisms $X' \to X$.

As in the classical situation this implies

Theorem (Log resolution)

For any integral logarithmic variety Z there exists a modification $Z_{\mathrm{res}} \to Z$ such that Z_{res} is log smooth. This is functorial w.r.t. log smooth morphisms $Z' \to Z$.

The method

In brief, we want to log-adjust all parts of the classical algorithm. The main adjustment is to only use log derivations:

(1) $\mathrm{logord}_x(I)$ is the minimal d such that $(D^{\log})^d(I_x) = \mathcal{O}_x$.

(2) Maximal contact is any $H = V(t)$, where t is any regular coordinate in $(D^{\log})^{d-1}(I_x)$ (in particular, H is toroidal).

(3) The coefficient ideal $C(I)$ is just $\sum_{i=0}^{d-1}((D^{\log})^i(I))^{d!/(d-i)}$.

(4) In addition, J is (I,d)-admissible if $I \subseteq J^d$ and, for appropriate coordinates, $J = (t_1, \ldots, t_l, m_1, \ldots, m_r)$ for any set of monomials. Then $X' = Bl_J(X)$ is toroidal and the d-transform $I' = (I\mathcal{O}_{X'})(J\mathcal{O}_{X'})^{-d}$ is defined. Note that J is submonomial – a monomial ideal on the log submanifold $V(t_1, \ldots, t_l)$).

Infinite log order

- Note that $\mathrm{logord}(t_i) = 1$ but $\mathrm{logord}(m) = \infty$. This is the main novelty that allows functoriality w.r.t. extracting roots of monomials (Kummer covers).
- As a price we have to do something special when $\mathrm{logord}(I) = \infty$, but this is simple: just start with blowing up the minimal monomial ideal I_{mon} containing I. For example, if $I = (\sum_{i \in \mathbb{N}^r} m_i t^i)$ then $I_{\mathrm{mon}} = (m_i)$. The single toroidal blow up makes logord finite! (This result is due to Kollár.)
- Our algorithm is simpler, in particular, it avoids both complications (max contact is given by a regular coordinate!).
- In a sense, we completely separate dealing with regular coordinates via log order and dealing with monomials via combinatorics (i.e. toroidal blow ups).
- The invariant is just $(d_n, \dots, d_1)$ with $d_i \in \mathbb{N}$, $d_1 \in \{0, \infty\}$.

Orbifolds

- Is all this so elementary? Where is the cheating?
- Well. Our algorithm does not distinguish monomials and their roots. In fact, we view this as a serious advantage (log smooth functoriality). As another side of the coin, to achieve correct weights and admissibility, the algorithm often insists to use Kummer monomials $m^{1/d}$.
- This can be by-passed by working on log-étale Kummer covers, which is ok due to the strong functoriality we prove. The Kummer-local description remains the same as we saw. However, in order to describe the algorithm via modifications of X we have to use orbifolds and non-representable modifications $X' \to X$ that we call Kummer blow ups.
- This is ok for applications, because we can remove the stacky structure afterwards by a separate torification algorithm. Though the latter is only compatible w.r.t. smooth morphisms.

An example

Example

(i) Take $X = \mathrm{Spec}\,\mathbb{C}[t, m]$ and $I = (t^2 - m^2)$. Then $\mathrm{logord}_O(I) = 2$, $H = V(t)$, $C(I)|_H = (m^2, 2)$, the order reduction of $C(I)|_H$ blows up $(m^2)^{1/2} = (m)$, and the order reduction of I blows up (t, m). Just as in the classical case.
(ii) If $I = (t^2 - m)$ then $\mathrm{logord}_O(I) = 2$, $H = V(t)$, $C(I)|_H = (m, 2)$, the order reduction of $C(I)|_H$ blows up $(m^{1/2})$, and the order reduction of I blows up $(t, m^{1/2})$. This is a non-representable Kummer blow up whose coarse moduli space $Bl_{(t^2,m)}(X)$ is not toroidal.

Remark

More generally, the weighted blow up of $((t_1, d_1), \ldots, (t_r, d_r))$ in $\mathbb{A}^n$ is the coarse space of a non-representable modification with a smooth source. They are used in the dream algorithm of [ATW19] and should be useful for other classical problems in birational geometry.