On the virtual Euler characteristics of the moduli spaces of semistable sheaves on complex projective surfaces

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Plan of talk:

1. Betti numbers of the moduli spaces of semistable vector bundles on a curve/coherent sheaves on a surface
2. Donaldson's polynomial invariants
3. Input from “supersymmetric” gauge theories
4. Virtual techniques from enumerative invariant theories

Introduction: the curve case

Consider $\mathcal{M}_C^{ss}(n, r)$ the moduli space of semistable vector bundles of rank r with degree n over a complex curve C. Assume for simplicity n and r are coprime to let $\mathcal{M}_C^{ss}(n, r)$ smooth.

- Harder–Narasimhan ('75) obtained an inductive formula of and hence computed the Poincaré polynomial of $\mathcal{M}_C^{ss}(n, r)$ via the Weil conjecture (Deligne,'74) by counting in a finite field.
- Atiyah–Bott ('82) reproved and also improved the result by Harder–Narashimhan via the infinite dimensional Morse theory.
- Kirwan ('83) obtained the Poincaré polynomial of GIT quotient as a finite-dimensional version of the result by Atiyah–Bott.

cf. Later, Earl–Kirwan ('00) obtained an inductive formula of the Hodge–Poincaré polynomial of $\mathcal{M}_C^{ss}(n, r)$ without using the Weil conjecture.

The surface case: not only a higher-dimensional analogue

Göttsche ('90) computed the Betti numbers of Hilbert schemes of points $X^{[n]}$ on a projective surface X. Subsequently, Yoshioka ('94) did it for $\mathbb{P}^2$ (cf. work by Klyachko ('91)), ruled surfaces ('95), and elliptic surfaces ('96) (both for the semistable=stable and $\mathrm{Ext}^2_0(E,E) = 0$ case).

A new feature emerges when one considers it on a complex projective surface, namely, one can consider the **generating series** of e.g. the Euler characteristics by varying c_2 (can not do this on a curve). Surprisingly, one often finds modular forms. E.g., Göttsche obtained:

$$\sum_{k=0}^{\infty} e(X^{[k]})q^k = (q^{-\frac{1}{24}}\eta(q))^{-e(X)},$$

where $\eta(q) := q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1-q^n)$ is the Dedekind eta function and $e(X)$ is the Euler characteristic of X.

The instanton moduli spaces

- X: closed, oriented, simply-connected, smooth four-manifold
- $P \to X$: principal $SO(3)$-bundle with the first Pontryagin class p_1 and the second Stiefel–Whitney class w_2.

Fix a Riemannian metric g on X, and consider the Hodge star operator $*_g$ on $\Lambda^2_X := (\Lambda^2 T^* X)$. This satisfies $*_g^2 = 1$, so Λ^2_X decomposes as $\Lambda^2_X = \Lambda^+_X \oplus \Lambda^-_X$.

Definition: A connection A on P is said to be an *anti-self-dual instanton*, if the curvature F_A of A satisfies $F_A^+ := \pi_+(F_A) = 0$, where $\pi_+ : \Gamma(\mathfrak{g}_P \otimes \Lambda^2_X) \to \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$ is the projection and $\mathfrak{g}_P$ is the adjoint bundle of P.

We denote by $\mathcal{A}_P$ the set of all connections on P and by $\mathcal{G}_P$ the set of all gauge transformations on P. We consider the *anti-self-dual instanton moduli space*:

$$M_{X,g}(w_2, p_1) := \{A \in \mathcal{A}_P \,:\, F_A^+ = 0\}/\mathcal{G}_P.$$

This is an oriented smooth manifold of expected dimensions for a generic choice of Riemannian metrics of X, if $b_X^+ > 0$, where b_X^+ is the number of positive eigenvalues of the intersection form on $H^2(X, \mathbb{Z})$.

Uhlenbeck compactification:

$$\overline{M}_{X,g}(w_2, p_1) := \coprod_{\ell=0}^{-p_1/4} M_{X,g}(w_2, p_1 + 4\ell) \times S^\ell X,$$

where $S^\ell X$ is the ℓ-th symmetric product of X. This is equipped with a natural topology, and it is a compact Hausdorff space.

Donaldson's polynomial invariants

Assume $b_X^+ > 0$. Denote by 2d the expected dimension of $M_{X,g}(w_2, p_1)$ and by A_d the symmetric algebra of degree d generated by $H_2(X)$ and $H_0(X)$ with $\alpha \in H_2(X)$ degree 1 and $p \in H_0(X)$ degree 2.

One can define $\bar{\mu} : H_p(X, \mathbb{Z}) \to H^{4-p}(\overline{M}_{X,g}(w_2, p_1), \mathbb{Z})$ for $p = 0, 2$ via a universal bundle on the moduli space. We then define Donaldson's polynomial invariant $q_{X,d} : A_d \to \mathbb{Z}$ of degree d by

$$q_{X,d}(\sigma_1, \ldots, \sigma_n, p^m) := \int_{[\overline{M}_{X,g}(w_2,p_1)]} \bar{\mu}(\sigma_1) \cup \cdots \cup \bar{\mu}(\sigma_n) \cup \bar{\mu}(p)^m,$$

where $\sigma_1, \ldots, \sigma_n \in H_2(X)$, $p \in H_0(X)$ and $d = n + 2m$. This is independent of the choice of Riemannian metrics of X, if $b_X^+ > 1$.

If $w_2 \neq 0$, then there is no trivial bundle in lower strata of the Uhlenbeck compactification, and the fundamental class $[\overline{M}_{X,g}(w_2, p_1)]$ is well-defined. However, if $w_2 = 0$, then there is a trivial bundle in a lower stratum of the compactification. In this case, we introduce the notion of *stable range* (e.g. one requires $-p_1$ is sufficiently large) to have a well-defined fundamental class $[\overline{M}_{X,g}(w_2, p_1)]$.

Blowup formula by Friedman–Morgan: Consider the blowup $\widehat{X} \to X$ at a point in X and denote by e its exceptional divisor. Then Friedman and Morgan prove:

$$q_{\widehat{X},d+1}(\sigma_1, \ldots, \sigma_d, e, e, e, e) = -2q_{X,d}(\sigma_1, \ldots, \sigma_d).$$

This can be used to define the Donaldson invariants for *unstable range*.

Fintsushel–Stern ('96) obtained a universal form of blowup formula, surprisingly, it is expressed in terms of modular forms.

Göttsche ('96) determined the wall-crossing term under the assumption that *Kotschick–Morgan conjecture* is true by an effective use of the blowup formula, and it turned out to be written in terms of modular forms. Also, as an application, the Donaldson invariants on $\mathbb{P}^2$ was determined.

Kronheimer–Mrowka ('95) proved the structure theorem for the Donaldson invariants. Denote by $q_X : \oplus_d A_d \to \mathbb{Q}$ the polynomial invariants. They considered the following Donaldson series:

$$\sum \frac{q_X(\Sigma^d)}{d!} + \frac{1}{2} \sum \frac{q_X(p\Sigma^d)}{d!}.$$

under the assumption of X *simple type*, and proved that this formal series can be written by a finite collection of *basic classes*.

Supersymmetric Yang–Mills theories

Witten ('88) reformulated the Donaldson invariants in terms of a topologically-twisted theory of $\mathcal{N} = 2$ super Yang–Mills theory.

Seiberg–Witten ('94) subsequently discovered the Donaldson series can be written in terms of the intersection form and a finite collection of Seiberg–Witten invariants (cf. the work of Kronheimer and Mrowka) which can be defined through the moduli spaces of pairs consisting of a $U(1)$-connection and a positive spinor on X, via a generalisation of electro-magnetic duality which is believed to exist in the theory.

Moore–Witten ('97) generalised the work by Seiberg–Witten to the case including wall-crossing phenomena and clarified more the origin of the modularity in the theory by means of the *u-plane*.

Vafa–Witten ('94) considered a more symmetric model: a topological twist of $\mathcal{N} = 4$ Super Yang–Mills theory.

Vafa–Witten equations: Let X be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X, where G is a connected Lie group. Fix a Riemannian metric on X. For $(A, B, C) \in \mathcal{A}_P \times \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X) \times \Gamma(\mathfrak{g}_P \otimes \Lambda^0_X)$, we consider the following equations:

$$d_A^* B + d_A C = 0, \ F_A^+ + [B, C] + [B.B] = 0,$$

where $[B.B] \in \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$.

These equations with a gauge fixing equation form an elliptic system with index always zero.

(T ('13, '15) Taubes ('13, '17), and others study the compactification problem for the moduli space of solutions to these equations, but it looks very difficult at the moment.)

Vafa and Witten's "invariant" has the following form:

$$VW := \chi(\overline{M}_{X,g}(w_2, p_1)) + \sum_{\xi \in M_{VW} \setminus M_{X,g}(w_2,p_1)} \varepsilon(\xi),$$

where χ is the "Euler characteristic", M_{VW} is the moduli space of solutions to the Vafa–Witten equations, and $\varepsilon(\cdot)$ is a "signed counting" of points in $M_{VW} \setminus M_{X,g}(w_2, p_1)$. They conjectued the generating series of this could be written in terms of modular forms. They checked it when $(B, C) \equiv 0$, in this case, one does not have the second term in the above and the generating series is that of the Euler characteristics of the instanton moduli spaces.

A far-reaching explanation of the modularity: Hiraku Nakajima constructs a representation of e.g the Heisenberg algebra on the homology of moduli space of the Hilbert scheme of points on a surface ('97), then the modularity follows since the generating series is the character of the representation. (see also Li–Qin ('98–'02) on a blowup formula for the virtual Hodge polynomials.)

Enumerative invariant theories using virtual techniques

- Virtual techniques had developed in Gromov–Witten theory such as by Li–Tian ('98) and Behrend–Fantechi ('97) in Algebraic Geometry. (cf. Li–Tian ('98), Fukaya–Ono ('99) in Symplectic Geometry)
- **Donaldson–Thomas invariants**: Consider the moduli space of semistable sheaves on a compact Calabi–Yau three-fold. This is proper, but it is obstructed in general. Richard Thomas ('00) brought the virtual technique to the problem to solve it, i.e. by constructing a *perfect obstruction theory* on the moduli space, then obtains the *virtual fundamental class* of the moduli space, when there are no strictly semistable sheaves in the moduli space.
- **Joyce–Song** ('12) generalise this to the case in which the moduli space may contain strictly semistable sheaves.

Mochizuki's Donaldson invariants on projective surfaces

The Hitchin–Kobayashi correspondence tells the instanton moduli space is the same as the moduli space of semistable sheaves on a complex projective surface. (cf. Algebraic Donaldson invariants.)

Takuro Mochizuki ('09) constructed a perfect obstruction theory on the moduli space. Then, **Donaldson–Mochizuki invariants** are defined as the integrations over the virtual fundamental class.

Mochizuki formulates the invariants where the moduli space may have strictly semistable sheaves, proves a weak wall-crossing formula for his invariants, and expresses them in terms of Seiberg–Witten invariants. These lead to the determination of the wall-crossing terms and a resolution of Witten's conjecture $D = SW$ on complex projective surfaces both by **Göttsche–Nakajima–Yoshioka** ('08, '11) both with analysis (blowup formulae) on the Nekrasov partition function.

Virtual Euler characteristics of the moduli spaces

We denote by $\mathcal{M}$ the moduli space of semistable sheaves on a complex projective surface X. This carries a perfect obstruction theory due to Mochizuki, and one has the virtual tangent bundle $T_{\mathcal{M}}^{vir} = R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})_0[1]$, where $\mathcal{E}$ is a universal sheaf on $X \times \mathcal{M}$ and $\pi : X \times \mathcal{M} \to \mathcal{M}$ is the projection.

Fantechi and Göttsche ('10) define the *virtual Euler characteristic* of $\mathcal{M}$ by

$$e^{vir}(\mathcal{M}) := \int_{[\mathcal{M}]^{vir}} c_{vd}(T_{\mathcal{M}}^{vir}).$$

Göttsche and Kool ('17) express this as a Donaldson–Mochizuki invariant, and form a conjecture that the generating series of the above virtual Euler characteristics of the moduli spaces could be written in terms of modular forms and Seiberg–Witten invariants. They verified it in many examples.

Vafa–Witten invariants on projective surfaces

The Hitchin–Kobayashi correspondence (by Álvarez-Cónsul and García-Prada ('03), T ('13)) tells the moduli space of solutions to the Vafa–Witten equations is the same as the moduli space $\mathcal{N}$ of pairs (E, ϕ) consisting of a coherent sheaf E and a morphism $\phi : E \to E \otimes K_X$ with a certain stability condition on a complex projctive surface X, where K_X is the canonical bundle of X.

Richard Thomas and the speaker ('17) constructed a symmetric perfect obstruction theory on the above moduli space $\mathcal{N}$.

There is a $\mathbb{C}^*$-action on $\mathcal{N}$ induced by the multiplication $\phi \mapsto \lambda\phi$ by $\lambda \in \mathbb{C}^*$. The fixed loci $\mathcal{N}^{\mathbb{C}^*}$ of this action is:

$$\mathcal{N}^{\mathbb{C}^*} \cong \mathcal{M}_X^{ss}(p, r) \sqcup \mathcal{M}^{Higgs},$$

where $\mathcal{M}_X^{ss}(p, r)$ is the moduli space of semistable sheaves of rank r with fixed Hilbert polynomial p on X.

The moduli space $\mathcal{N}$ is not proper, but these fixed loci are proper, so we define:

$$VW := \int_{[\mathcal{N}^{\mathbb{C}^*}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{N}^{\mathbb{C}^*}})}.$$

By a result of **Jiang–Thomas** ('14), one sees:

$$\begin{aligned} VW &:= \int_{[\mathcal{N}^{\mathbb{C}^*}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{N}^{\mathbb{C}^*}})} \\ &= \int_{[\mathcal{M}^{ss}_X(p,r)]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{M}^{ss}_X(p,r)})} + \int_{[\mathcal{M}^{Higgs}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{M}^{Higgs}})} \\ &= \underbrace{\int_{[\mathcal{M}^{ss}_X(p,r)]^{vir}} c_{vd}(T^{vir}_{\mathcal{M}^{ss}_X(p,r)})}_{e^{vir}(\mathcal{M}^{ss}_X(p,r))} + \underbrace{\int_{[\mathcal{M}^{Higgs}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{M}^{Higgs}})}}_{\text{the contribution from the Higgs fields}} . \end{aligned}$$

This does resemble the form that Vafa and Witten envisaged, which we discussed earlier in this talk. In fact, this matches with the conjecture by Vafa and Witten.

For the K3 surface case, Thomas and the speaker ('17) obtained the following, assuming a conjecture by Yukinobu Toda ('11) in Donaldson–Thomas theory (it was later proved by Maulik and Thomas ('18)), which also matches with the conjecture by Vafa and Witten:

$$\sum_{c_2} VW_{r,c_2} q^{c_2} = \sum_{d|r} \frac{d}{r^2} \sum_{j=0}^{d-1} \eta\left(e^{\frac{2\pi i j}{d}} q^{\frac{r}{d^2}}\right)^{-24}.$$

To be precise, the above is for the $SU(r)$ Vafa–Witten invariants, namely, we fix the first Chern class of sheaves and consider ϕ with $\mathrm{tr}\phi = 0$, let us denote them by $VW^{SU(r)}_{r,c_2}$. Also, the *Vafa–Witten partition function* is defined to be:

$$Z^{SU(r)}_{VW}(q) := r^{-1} q^{\lambda} \sum_{c_2} q^{\frac{1}{2r}\mathrm{vd}} (-1)^{\mathrm{vd}} VW^{SU(r)}_{r,c_2},$$

where $\lambda := -\frac{1}{2}\chi(\mathcal{O}_X) + \frac{r}{24}K_X^2$, and vd is the virtual dimension of the moduli space $\mathcal{M}^{ss}_X(p,r)$.

More recently, **Jiang and Kool** ('20) proved the *S-duality conjecture* for $K3$ surfaces by constructing the Langlands dual side of the story, when the rank r is prime:

$$Z_{VW}^{SU(r)}(-1/\tau) = (-1)^{(r-1)\chi(\mathcal{O}_X)} \left(\frac{r\tau}{i}\right)^{-\frac{e(X)}{2}} Z_{VW}^{{}^L SU(r)}(\tau),$$

where $q = e^{2\pi i \tau}$, and the ${}^L SU(r)$ Vafa–Witten partition function $Z_{VW}^{{}^L SU(r)}$ on the right-hand side is defined through the moduli spaces of twisted sheaves by Yoshioka.

Most recently, Göttsche–Kool–Laarakker announced that they found possible closed formulae for the rank 4 and 5 (vertical) Vafa–Witten partition function. Interestingly, e.g. the rank 5 case involves **Rogers–Ramanujan's continued fraction**. They then claim an interesting new conjecture on the virtual Euler characteristics of the moduli spaces of semistable sheaves on a surface for the rank 4 and 5 cases by using the S-duality argument.