Fano varieties: From $D^b(X)$ to geometry

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Stanford, March 5, 2021

Fano varieties: $D^b(X)$

① ~~Kuznetsov components~~ → Geometry

② Stability conditions & wall-crossing

③ Brill-Noether constructions

X = smooth projective Fano variety

Bondal-Orlov: X can be recovered from $D^b(X)$

Question: How to make this effective?

Bridgeland '07: Stability conditions on $D^b(X)$, Wall-crossing

- Very powerful & effective for surfaces
  [Birational Geometry of Hilbert schemes, ..., Brill-Noether questions]
- But: · for $\dim X \geq 3$: hard to construct, wall-crossing hard to control
  Even for $\mathbb{P}^3$: only few examples understood
  (Schmidt, Bingyu Xia, Gallardo-Lozano Huerta-Schmidt, Schmidt, Rezaee)
  · Only few more systematic results (Feyzbakhsh-Thomas, Toda, Schmidt-Sung)

1. **Kuznetsov:** Consider admissible subcategory $Ku(X) \xrightarrow{i} D^b(X)$

Examples: • X = Fano threefold $\mathrm{Pic}(X) = \mathbb{Z}\cdot H$

(A) index two: $-K_X = 2\cdot H$ $\quad D^b(X) \supset Ku(X) = \mathcal{O}^\perp \cap \mathcal{O}(1)^\perp$

5 families Y_d $\quad 1 \le d = H^3 \le 5$ $\quad \mathcal{O}^\perp = \{E \mid \mathrm{Hom}^\cdot(\mathcal{O}, E) = 0\}$

(B) index one: $-K_X = H$ $\quad H^3 = 2g-2$ $\quad 2 \le g \le 12$ $\quad g \ne 11$

10-families X_{2g-2}

Mukai: $\exists!$ stable rigid $\mathcal{E}_2$ of rk $=2$, $c_1 = -H$

g even, $g \ge 6$: $\quad Ku(X) := \mathcal{E}_2^\perp \cap \mathcal{O}^\perp$

(C) $X \subseteq \mathbb{P}^5$ cubic 4-fold $\quad Ku(X) = \mathcal{O}^\perp \cap \mathcal{O}(1)^\perp \cap \mathcal{O}(2)^\perp$

(D) X GM 4-fold e.g. $X \subseteq Gr(2,5)$ (2,1)-complete intersection

$$Ku(X) = \mathcal{U}^\perp \cap \mathcal{O}^\perp \cap \mathcal{U}(1)^\perp \cap \mathcal{O}(1)^\perp$$

$Ku(X)$ better controlled, e.g.

- Y_5, X_{22}: D^b(quiver)
- $Y_4, X_{12}, X_{16}, X_{18}$: D^b(curve)
- Y_2, X_{10}: Enriques category
  $S = i \circ [2]$ i = involution

Cubic 4fold, GM 4-fold:
K3 category
$\mathrm{Hom}(E,F) = \mathrm{Hom}(F, S_{Ku(X)} E)^\vee$
$S_{Ku(X)} = [2]$

A. Perry '20: Hodge structure $\hat{H}(Ku(X))$ associated to $Ku(X)$

X = Fano 3-fold $\hat{H}(Ku(X)) = H^3(X) \rightsquigarrow$ intermediate Jacobian

GM / cubic 4-fold $\hat{H}(Ku(X)) \longleftrightarrow H^4(X)$

Torelli question: Does $H^*(X)$ determine X? Breaks into:

- Does $H^*(X) \supseteq \hat{H}(Ku(X))$ determine $Ku(X)$?
- Does $Ku(X)$ determine X?

Projection functors: $i^*, i^! : D^b(X) \longrightarrow Ku(X) \xrightarrow{i} D^b(X)$

$Ku(Y_3) = \mathcal{O}^\perp \cap \mathcal{O}(1)^\perp$

Very natural: E.g. ① conic $C \subset Y_3$, $Y_3 \subset \mathbb{P}^4$ cubic 3fold

$\langle C \rangle = \mathbb{P}^2$ $\langle C \rangle \cap Y_3 = C \cup L$

$i^* \mathcal{O}_C(1) = ?$ $\mathcal{O}(1) \to \mathcal{O}_C(1) \to I_C(1)[1]$

$\mathcal{O}^{\oplus 2} \to I_C(1) \to E$ $\rightsquigarrow$ $\mathcal{H}^{-1}(E) = \mathcal{O}(-1)$

$\mathcal{H}^0(E) = \mathcal{O}_L(-1)$

$\rightsquigarrow i^*_{\mathcal{O}_C(1)} = E[1] = I_L^\vee(-1)[2]$

② smooth twisted cubic $C \subset Y_3$

$\langle C \rangle \cap Y_3 = S_3$ cubic surface $S_3 \xrightarrow{\pi} \mathbb{P}^2$

$i^* \mathcal{O}_C(1) = ?$ $i^*\mathcal{O}_C(1)$

$\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$

$0 \to I_C(1) \to I_{C/S_3}(1)$

$C \to |C|$ on S_3 $\rightsquigarrow$ $\mathbb{P}^2$-fibration

② Stability conditions on $Ku(X)$

Consists of:
- $\mathcal{A} \subset Ku(X)$ abelian subcategory "heart of a bounded t-structure"
- rank-function and degree-function on $\mathcal{A}$ $\rightsquigarrow$ $\mu = \frac{\deg}{rk}$

B-Lahoz-Macrì-Stellari '17: Construction

BLM-Nuer-Perry-S '19: in families; moduli spaces of semistable objects

P-Perfusi-Zhao '20: GM 4-folds

Construction: For 3-folds: auxiliary "weak stability" on $D^b(X)$

$\rightsquigarrow$ restrict it to $Ku(X)$

For cubic/GM 4-folds: $Bl_\ell X \longrightarrow \mathbb{P}^3$ $\boxed{Ku(X) \subset D^b(\mathbb{P}^3, \mathcal{B}_0)}$

conic fibration

Examples:

$Y_3 \subset \mathbb{P}^4$ cubic (Bernardara-Macrì-Mehrotra-Stellari '09)

$\rightsquigarrow F(Y_3)$ = Fano variety of lines as a mod. space in $Ku(Y_3)$

Cor: $Ku(Y_3) \rightsquigarrow Y_3$

Y_d, $d \neq 3$ (Altavilla-Rota-Petkovic '19):

$\boxed{i^*\mathcal{O}_y \text{ stable for } y \in Y_d}$

e.g. $Y_2 \xrightarrow{2:1} \mathbb{P}^3$

$M_\sigma([i^*\mathcal{O}_y]) = Y_2 \cup \mathcal{C} \leftarrow$ 3-dim'l

$Y_2 \cap \mathcal{C}$ = quartic K3 = branch locus

5dim'l p.p.a.v.

$H^3(Y_3)$

Y_3 (B-Beentjes-Feyzbakhsh-Hein-Martinelli-Schmidt '20):

$Y_3 \subset M_\sigma([i^* O_y]) = $ desingularization of $\Theta \subset J(Y_3)$

$\Downarrow$

0

Cor: reproves both categorical & classical Torelli

X_{10} (Shizhou Zhang '20): studied all 3-dim'l moduli spaces

Cor: $Ku(X_{10}) \neq Ku(Y_2)$ contradicting Conjecture by Kuznetsov.

$X \subset \mathbb{P}^5$ cubic 4 fold: $\widetilde{H}^{1,1}(Ku(X)) = A_2$ X very general

$\lambda_2 \nwarrow \searrow \lambda_1$

$\lambda_1^2 = \lambda_2^2 = 2$ $(\lambda_1, \lambda_2) = -1$

Li-Pertusi-Zhao:

$M_\sigma(\lambda_1) = F(X) = $ Fano 4-fold of lines on X

$M_\sigma(\lambda_2 - \lambda_1) = $ LLSvS 8-fold and $x \mapsto i^* O_x$ gives $X \overset{\text{Lagrangian}}{\subset} M_\sigma(\lambda_2 - \lambda_1)$

$M_\sigma(2\lambda_1)$: singular O'Grady model of Voisin 10-fold

③ Brill-Noether

Ex.: $D^b(X_{2g-2}) = \langle Ku(X_{2g-2}), \mathcal{E}_2, \mathcal{O} \rangle$

$\mathcal{O}^\perp = \langle Ku(X_{2g-2}), \mathcal{E}_2 \rangle = \{ V, F, \phi \mid V \in D^b(k), F \in Ku(X_{2g-2}), \phi : \underbrace{i^! \mathcal{E}_2 \otimes V \to F}_{} \}$

determined by $Ku(X_{2g-2}) + i^! \mathcal{E}_2$!

Now: X smooth cubic 4-fold — B-Bertram-Macrì-Perry

X Hassett-special $\Leftrightarrow$ $\mathbb{Z} \cdot [X \cap \mathbb{P}^3] \subsetneq H^{2,2}(X, \mathbb{Z})$

$\Leftrightarrow$ $A_2 \subsetneq \widetilde{H}^{1,1}(Ku(X), \mathbb{Z})$

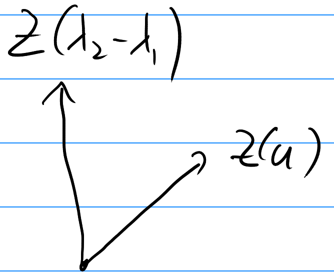
$X \in \mathcal{C}_d$ iff exists rk 3-lattice $A_2 \subset \Lambda \subset \widetilde{H}^{1,1}(Ku(X), \mathbb{Z})$ with $\mathrm{discr}(\Lambda) = -d$

Can we make this geometric ?

Prototype: $X \in \mathcal{C}_8 \Leftrightarrow$ X contains a plane

$v \in \tilde{H}_1^{1,1}(Ku(X_\cdot),\mathbb{Z})$, primitive, $v^2 \geq -2 \Rightarrow M_\sigma(v) \neq \emptyset$ & smooth proj. HK variety of dim v^2+2.

Consequences for X?



Idea: Pick $u \in \tilde{H}^{1,1}(Ku(X),\mathbb{Z})$, $u^2 \geq -2$, $u \notin A_2$, $\boxed{(u, \lambda_2-\lambda_1)=1}$.

$E \in M_\sigma(u)$, $F \in M_\sigma(\lambda_2-\lambda_1) \Rightarrow \chi(E,F) = -1$

$M_\sigma(\lambda_2-\lambda_1) \ni BN_E = \{F \mid \hom(E,F) > 0\}$ has expected codim 2

$\cup$

$X \rightsquigarrow \Sigma := X \cap BN_E$ $\qquad (u,\lambda_1) = a$

Conjecture: For $X \in \mathcal{C}_d$ very general, $\boxed{d = 6a^2+6a+2-3(u,u)}$
this defines a surface $\Sigma \subset X$

$H^2.\Sigma = \frac{3}{2}a^2 + \frac{3}{2}a + 1 \qquad \Sigma.\Sigma = \frac{1}{3}\deg(\Sigma)^2 - d$

Case $a^2=0$:

$M_\sigma(a) = S$ polarised K3 surface, deg $\boxed{6a^2+6a+2}$

$M_\sigma(\lambda_2-\lambda_2) \dashrightarrow S^{[4]}$, $s \in S$

$\cup$

$BN_s = \{Z \mid s \subset Z\}$

Theorem (BBMT): $X \in \mathcal{C}_{6a^2+6a+2}$ very general, $a>1$:

① $M_\sigma(\lambda_2-\lambda_1) = S^{[4]}$ (nef cone = movable cone)

② $X \cap BN_s$ = surface Σ_a of degree $1+\frac{3}{2}a+\frac{3}{2}a^2$

Question: Does this shed any light on rationality of X?

Observation 1: For $d=14$, $d=38$: These surfaces appear in rationality constructions ($d=38$: Russo – Staglianò)

Observation 2: Which rational map is resolved by Blow-up at BN_E?

Ex: $$S^{[4]} \dashrightarrow S^{[5]} \qquad Z \mapsto Z \cup s$$
$$\begin{array}{c} S^{[4]} \\ \cup \\ BN_s \end{array}$$

Cor: $X \in \mathcal{C}_{6a^2+6a+2}$ very general $\Rightarrow Bl_\Sigma X \subset S^{[5]}$

Now: Run MMP for $S^{[5]}$ (BM '14)

& look at proper transforms for $Bl_\Sigma X$:

Ex: $d = 38$

$$\begin{tikzcd} S^{[5]} \arrow[dr] & & M_1 \arrow[dl] \arrow[dr] & \\ & \overline{M}_1 & & \overline{M}_2 \end{tikzcd}$$

$\mathbb{P}^3$-bdl over 4-fold

$\mathbb{P}^4$-bdl over S

$$
\begin{tikzcd}
\mathrm{Bl}_{\Sigma} X \arrow[dr] & & X_1 \arrow[dl] \arrow[dr] & \\
& \bar{X}_1 & & \bar{X}_2 = \mathbb{P}^4
\end{tikzcd}
$$

contracts lines 3-secant to Σ

contracts conics 5-secant to Σ

(Not an independent proof yet....)