

# GROTHENDIECK'S LOCALIZATION PROBLEM

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Let  $\mathbf{P}$  be a property of noetherian local rings.

**Definition.** Let  $k$  be a field, and let  $X$  be a locally noetherian  $k$ -scheme. We say that  $X$  is *geometrically  $\mathbf{P}$*  over  $k$  if the local rings of  $X \otimes_k k'$  are  $\mathbf{P}$  for every finite field extension  $k \subseteq k'$ .

**Definition.** Consider a flat morphism  $f: Y \rightarrow X$  of locally noetherian schemes. We say that  $f$  is *geometrically  $\mathbf{P}$*  if  $f^{-1}(x)$  is geometrically  $\mathbf{P}$  over  $\kappa(x)$  for every  $x \in X$ .

**Permanence conditions** [EGAIV<sub>2</sub>, §7]. Denote by  $\varphi: (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$  a local flat map of noetherian local rings.

- (I) (Ascent) If  $A$  is  $\mathbf{P}$  and  $\varphi$  is geometrically  $\mathbf{P}$ , then  $B$  is  $\mathbf{P}$ .
- (II) (Descent) If  $B$  is  $\mathbf{P}$ , then  $A$  is  $\mathbf{P}$ .
- (III) (Lifting from Cartier divisors/Inversion of adjunction) If  $A/t$  is  $\mathbf{P}$  for some nonzerodivisor  $t \in \mathfrak{m}$ , then  $A$  is  $\mathbf{P}$ .
- (IV) (Localization) If  $A$  is  $\mathbf{P}$ , then  $A_{\mathfrak{p}}$  is  $\mathbf{P}$  for every prime ideal  $\mathfrak{p} \subseteq A$ .

**Grothendieck's localization problem** (see [EGAIV<sub>2</sub>, Rem. 7.5.4(i)]). *Consider a flat local map  $\varphi: (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$  of noetherian local rings. If both  $\mathrm{Spec}(\widehat{A}) \rightarrow \mathrm{Spec}(A)$  and  $\mathrm{Spec}(B \otimes_A k) \rightarrow \mathrm{Spec}(k)$  are geometrically  $\mathbf{P}$ , then is  $\mathrm{Spec}(\varphi)$  geometrically  $\mathbf{P}$ ?*

**Theorem** [Mur, Thm. B]. *Grothendieck's localization problem holds for every property  $\mathbf{P}$  such that regular implies  $\mathbf{P}$ , and such that  $\mathbf{P}$  satisfies (I)–(IV).*

$\mathbf{P}$	Theorem	Ascent (I)	Descent (II)	Deformation (III)	Localization (IV)
regular	[And74, Thm. on p. 297]	[Mat89, Thm. 23.7]		[EGAIV <sub>1</sub> , Ch. 0, Cor. 17.1.8]	[Mat89, Thm. 19.3]
normal	[Nis81, Prop. 2.4]	[Mat89, Cor. to Thm. 23.9]		[Sey72, Prop. I.7.4]	[Bou98, Ch. V, §1, n° 5, Prop. 16]
weakly normal	[Mur, Cor. 4.12]	[Kol16, Thm. 37]	[Man80, Cor. II.2]	[Mur, Prop. 4.11]	[Man80, Cor. IV.2]
seminormal	[Mur, Cor. 4.12]	[Kol16, Thm. 37]	[GT80, Thm. 1.6]	[Hei08, Main Thm.]	[GT80, Cor. 2.2]
reduced	[Nis81, Prop. 2.4]	[Mat89, Cor. to Thm. 23.9]		[EGAIV <sub>2</sub> , Prop. 3.4.6]	[Bou98, Ch. II, §2, n° 6, Prop. 17]
complete intersection	[Tab84, Thm. 2]	[Avr75, Thm. 2]		[BH98, Thm. 2.3.4(a)]	[Avr75, Cor. 1]
Gorenstein	[HS78, Thm. 3.3; Mar84, Thm. 3.2]	[BH98, Cor. 3.3.15]		[BH98, Prop. 3.1.19(b)]	[BH98, Prop. 3.1.19(a)]
Cohen–Macaulay	[AF94, Thm. 4.1]	[BH98, Thm. 2.1.7]		[BH98, Thm. 2.1.3(a)]	[BH98, Thm. 2.1.3(b)]
Cohen–Macaulay + $F$ -injective	[Mur, Cor. 4.3]	[Ene09, Thm. 4.3]	[Has10, Lem. 4.6]	[Fed83, Thm. 3.4(1)]	[Has10, Cor. 4.11]
$F$ -rational	[Mur, Cor. 4.5] <sup>e</sup>	[Ene00, Thm. 2.27] <sup>e</sup>	[DM, Prop. A.5]	[HH94, Thm. 4.2(h)] <sup>CM</sup>	[HH94, Thm. 4.2(f)] <sup>CM</sup>
pseudo-rational	[Mur, Cor. 4.17] <sup>char0</sup>	[Elk78, Thm. 5] <sup>char0</sup>	[Mur, Prop. 4.15] <sup>rc</sup>	[Elk78, Thm. 2] <sup>char0</sup>	[LT81, §4, Cor. of (iii)] <sup>rc</sup>

<sup>e</sup> This holds for excellent rings.

<sup>CM</sup> This holds for homomorphic images of Cohen–Macaulay rings.

<sup>rc</sup> This holds for rings that have residual complexes.

<sup>char0</sup> This holds for rings essentially of finite type over fields of characteristic zero.

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