# Algebraic hyperbolicity and Lang-type loci in hypersurfaces

joint w/ Eric Riedl

**Mordell's Conjecture / Faltings' Thm**

$X$ smooth proj curve of genus $\geq 2$ defined / # field $k$.

Then $|X(k)| < \infty$.

Lang - How do we generalize to varieties of higher dimension?

- $X$ is of general type

  Blow-up rational point to introduce $\infty$ly many rat'l pts.

- $\Omega_X$ is ample

- $X$ has a hyperbolic metric with constant negative curvature
- $X$ is uniformized by $\Delta$.

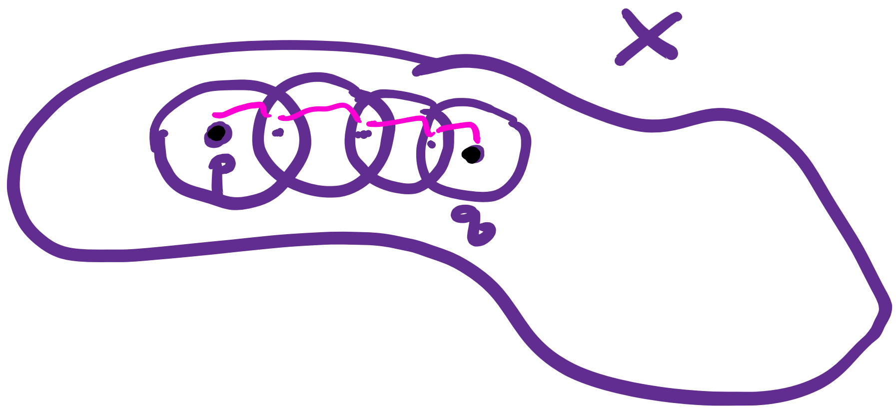
  There exists no nonconstant maps

  $$\mathbb{C} \longrightarrow X.$$

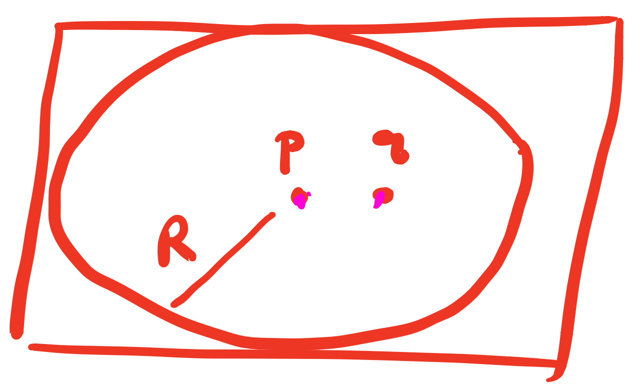
Lang - in generalizing to higher dim highlights <u>hyperbolicity</u>.

$X$ is <u>hyperbolic</u> if Kobayashi pseudo-metric is nondegenerate.

Kobayashi pseudo-metric




$\mathbb{C}$ is not hyperbolic.




$R \to \infty$

$d(p,q) < \varepsilon$.

If $f : \mathbb{C} \to X$ is nonconstant, then $X$ is not hyperbolic.

Hyperbolic $X$ does not contain rational or elliptic curves.

$X$ is <u>Brody-hyperbolic</u> if every map $f: \mathbb{C} \longrightarrow X$ is constant.

<u>Brody</u>: For <u>compact</u> complex manifolds

hyperbolic = Brody hyperbolic

For our purposes, Kobayashi and Brody hyperbolicity coincide.

[They do not coincide for noncompact complex manifolds.]

<u>Lang Conjectures:</u> $X$ variety of general type. There exists proper alg. set $Z$ st.

(i) The images $f: \mathbb{C} \longrightarrow X$ nonconst. lie in $Z$.

(ii) Images of nonconstant maps from rational curves and abelian varieties lie in $Z$.

(iii) The complement of $Z$ is hyperbolic.

(iv) If $X$ is defined over $k$

$$|X \setminus Z\,(k)| < \infty.$$

It is <u>hard</u> to check hyperbolicity.

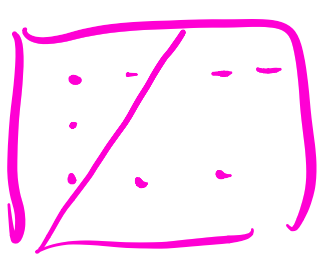
Demailly introduced an algebraic version.

A projective variety $X$ is <u>algebraically hyperbolic</u> if $\exists\ \varepsilon > 0$ st $\forall$ curves $C$

$$2g(C) - 2 \geq \varepsilon \ \underline{\deg C}$$

$\uparrow$ geometric genus.

If $X$ is algebraically hyperbolic, then $X$ does not contain rational or elliptic curves.



$$\mathbb{C}^n / \Lambda \approx T$$

Hyperbolic $\Longrightarrow$ algebraically hyperbolic

Demailly conjectures the converse
for smooth projective varieties.

It is still hard, but easier to check
algebraic hyperbolicity.

Two types of resuts:

1. Verifying hyperbolicity
2. Identifying Lang loci.

For hypersurfaces in $\mathbb{P}^n$

Ein, Clemens-Ran, Pacienza, Voisin

[ Siu, Demailly ..... ]

**Thm**. $n \geq 4$, $X \subset \mathbb{P}^n$ very general hypersurface of degree $d \geq 2n-2$, then $X$ is algebraically hyperbolic.

This is best possible! If $d \leq 2n-3$, $X$ contains lines.

Geng Xu building on work of Ein.

**Thm** (Xu) $\quad n = 3$ $\quad d=5$

$$2g(C) - 2 \geq (d-5) \deg C$$

If $d \geq 6$, $X$ is algebraically hyperbolic.

Main remaining case quintics in $\mathbb{P}^3$

**Thm.** (C, Riedl) Let $X \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 5$. A curve of degree $dk$ satisfies

$$2g(C) - 2 \geq dk(d-5) + k$$

In particular, a very general quintic is algebraically hyperbolic with $\varepsilon = \frac{1}{5}$.

In a different direction, Haase and Ilten started the classification of algebraically hyperbolic surfaces in toric 3-folds.

<u>Example Thm</u> (C, Riedl)

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

① $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ a very general surface of class $a_1 H_1 + a_2 H_2 + a_3 H_3$ is algebraically hyperbolic

$\iff$ (up to permutations)

$$a_1, a_2, a_3 \geq 3 \quad \text{or} \quad a_1 = 2,\ a_2, a_3 \geq 4$$

② $\mathbb{P}^2 \times \mathbb{P}^1$ a very general surface of class $a H_1 + b H_2$ is algebraically hyperbolic $\iff$

$$a \geq 4,\ b \geq 3 \quad \text{or} \quad b = 2,\ a \geq 5$$

(More generally, can do $\mathbb{F}_e \times \mathbb{P}^1$)

③ Blow up$_p$ $\mathbb{P}^3$ a very general surface of class $aH - bE$ is alg. hyp.

$\iff$

$a \geq b+2,\ b \geq 4$ or $b=0,\ a \geq 5.$

## Ingredients:

**Obs 1:**

$$C \xrightarrow{h_t} X$$

$$0 \to T_C \longrightarrow h^* T_X \longrightarrow N_h \to 0$$

$$\underbrace{\deg N_h}_{} = 2g(C) - 2 - K_X \cdot C$$

Bound the degree of normal bdl. from below.

**Step 2.** Get positivity from universal family

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{P}^3 \\ \downarrow & & \\ H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \setminus 0 & & \end{array}$$

Suppose each surface contains a curve of degree $e$ and genus $g$.

After étale base change

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] & \mathcal{X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{P}^3 \\ & \mathcal{U} & \end{tikzcd}$$

$t \in \mathcal{U}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\mathcal{X}$

$$N_{h_t} \simeq N_h|_{C_t}$$

<u>Step 3</u>. Understand $N_h$.

$$0 \to T_{\mathcal{C}|\mathbb{P}^3} \to T_{\mathcal{X}|\mathbb{P}^3} \to \boxed{\mathcal{K}} \to 0$$

$\mathcal{K} \hookrightarrow N_h$ torsion cokernel.

$$\deg(N_{h_t}) \geq \deg(\mathcal{K}|_{C_t})$$

<u>Step 4</u>. Use appropriate Lazarsfeld-Mukai bundles to quantify positivity

$M_1$

of $\mathcal{K}$.

$$0 \to M_d \to \mathcal{O}_{\mathbb{P}^n} \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0$$

$$T_{\mathcal{X}/\mathbb{P}^n} \simeq \pi_2^* M_d$$

So we need to understand +vity of $M_d$

Voisin-Pacienza-Clemens-Ran game:

$$\bigoplus^{s} h^* \pi_2^* M_1^{\mathbb{P}^n} \twoheadrightarrow \mathcal{K} \quad \text{for some } s$$

Find small $s$ $\quad \Omega_{\mathbb{P}^n}(1)$

$$n=3 \quad h_t^* \pi_2^* \Omega_{\mathbb{P}^3}(1) \longrightarrow N_{h_t/X_t}$$

with torsion cokernel.

Step 5: $\quad C \xrightarrow{f} \mathbb{P}^n$ gen. inj.

map from curve of deg $e$. Line bdl

quotients of $f^* \Omega_{\mathbb{P}^n}(1)$ of deg $-m$

give rise to surface scrolls of degree at most $e-m$. containing $f(C)$

quotient Euler seq.

$$\begin{tikzcd}
 & S \arrow[d] \arrow[rr, bend left] & & & & \\
0 \arrow[r] & f^*\Omega_{\mathbb{P}^n}(1) \arrow[r] \arrow[dr] & \mathcal{O}_C^{n+1} \arrow[r] \arrow[drr] & \mathcal{O}_C(1) \arrow[r] & 0 \\
 & & Q & & Q' &
\end{tikzcd}$$

$\mathbb{P}Q' \longrightarrow \mathbb{P}^n$ gives the scroll.

<u>Step 6</u> $\quad h^*\Omega_{\mathbb{P}^3}(1) \longrightarrow N_{h/X}$

$Q$ the image

$$\begin{array}{c} \deg N_{h/X} \geq \deg Q \geq k - dk \\ \| \\ dk(4-d) + 2g-2 \end{array}$$

$$\boxed{2g-2 \geq dk(d-5) + k}$$

$\blacksquare$

# Questions:

① $X$ very general hypersurface

What are the possible genera of curves? What are the gaps?

② $C \subset X$ a curve.

Find the scroll that contains $C$ for which the degree of the universal line is minimal.

Is the cone/$C$ with vertex the most singular point optimal?

③

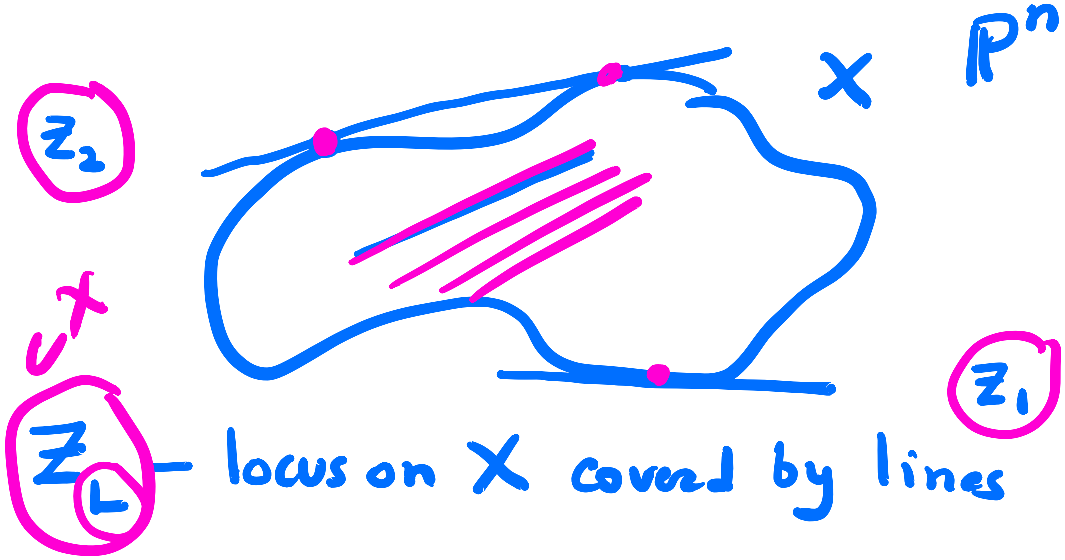
What about Lang type conjectures for algebraic hyperbolicity?

$X \subset \mathbb{P}^n$ is a hypersurface

$n+1 < d < 2n-2$, then $X$ contains lines.

$X$ is not algebraically hyperbolic.

Can we characterize the Lang locus?



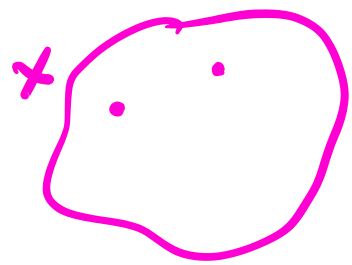
$Z_i$ - the locus on $X$ swept out by lines that intersect $X$ in at most $i$-distinct points.

deg. d.

**Theorem** (C, Riedl) $X \subset \mathbb{P}^n$ v.g. hypersurface.

① If $d \geq \boxed{\frac{3n+2}{2}}$, then any curve not lying in $Z_2$ satisfies

$$2g(C) - 2 \geq \deg C$$




② If $d \geq \frac{3n}{2}$, then $X$ contains lines but no other rational curves.

③ Let $k > 0$ be an integer. If $d \geq \frac{3n+1-k}{2}$, then the only points of $X$ rationally equivalent to a $k$-dim family of points lie in $Z_1$.

④ If $d \geq \frac{3n+3}{2}$, then any point on $X$ rationally equivalent to another point of $X$ lies in $\underline{Z_2}$.

The key to these theorems is a Grassmannian technique.

$B \subset G(k-1, n)$ irred family

the containing family of $B$

$C \subset G(\underline{k}, n)$ is the family of $c$ st $c \supset b$ for $b \in B$.

- Riedl-Yang technique:

If $B \subsetneq G(k-1, n)$, then

$\operatorname{codim} C \leq \operatorname{codim} B - 1.$

Thm (C, Riedl) If codim $B \geq 2$ and codim $C$ = codim $B - 1$, then $\exists$ irreducible variety $Z \subset \mathbb{P}^n$ of dimension $n-k+1-\varepsilon$ st. $B$ is the set of $(k-1)$-planes intersecting $Z$.

More generally $B \subset G(k-1, n)$ is $j$-clustered if codim $C$ = codim $B - j$

We don't know how to characterize these families.

- $[B] = \sum a_\lambda \sigma_\lambda$

  if $a_\lambda \neq 0 \Rightarrow$ length $\lambda \leq j$.

- codim $B \leq j(n-k+1)$

- In case of $=$, $B$ parametrizes $(k-1)$-dim linear spaces that contain a fixed $\mathbb{P}^{j-1}$.

Idea: Start with a universal pointed hypersurface of degree d in $\mathbb{P}^n$

Pass to hyperplane sections

- Either the codim improves by $\geq 2$
- Lines explain the failure of improvement.

Eg. Roitman - A very general point of a CY variety is rationally equivalent to only fin. many pts.

$$B_{r,d} \subset U_{r,d}$$

$(p, X)$ st $p$ is rat'l equiv to at least 1-dim family of pts.

codim $B_{d-1,d} \subset U_{d-1,d} \geq 1$.

either $B_{d-c,d} \subset U_{d-c,d}$ has codim

$2c+1$ or you find lines.