Fano varieties associated to hyperkähler manifolds

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I. Fano varieties

X Fano if $-K_X$ ample

$\boxed{\text{Ex:}}$ $\mathbb{P}^n$, $X_d \subset \mathbb{P}^n$ $d < n+1$, $X_{d_1,\dots,d_k} \subset \mathbb{P}^n$

(Kollár-Miyaoka-Mori) Only fin. many def. classes of sm. Fano varieties of a given dim

$\Rightarrow$ Attempts to classify (Birational geometry)

$\boxed{\text{Ex:}}$ $\dim X \leq 3$

- $\boxed{\text{Index}(X)}$ = largest $r \geq 1$ st $K_X \equiv -rH$ ample div on X

Complete class. $\rho = 1$, index $\geq \dim X - 2$

$\rho > 1$ index $\geq \frac{1}{2}(\dim X + 1)$

$\boxed{\text{Takeaway:}}$ **Low index** $\Rightarrow$ harder to classify

II. Hyperkähler manifolds

Motivation: S K3 surface

$\shortparallel$

$\mathbb{C}$-kähler surface, $\pi_1(S) = 0$, $H^0(S, \Omega^2_S) = \mathbb{C}\omega$

Option 1: X dim n, $H^0(X, \Omega^n_X) = \mathbb{C}\omega$ Calabi-Yau

Option 2: X dim, $\mathbb{C}$-kähler, $\pi_1(X) = 0$, $H^0(X, \Omega^2_X) = \mathbb{C}\omega$

Hyperkähler manifold

$\Rightarrow K_X \equiv 0 \Rightarrow X$ is Calabi-Yau

& $\dim X$ is even

$\boxed{\text{Thm:}}$ (Beauville-Bogomolov)

[illegible]

X compact Kähler, $c_1(X)=0$

$\exists$ finite étale cover $\pi : \prod M_i \to X$ st each M_i one of: $\mathbb{C}$-tori, CY, HK

Nice Properties

K3 surfaces	HK manifolds
① $H^2(S,\mathbb{Z})$ has int. form $\Rightarrow$ lattice theory	① $H^2(X,\mathbb{Z})$ has integral quad. form (Beauville–Bogomolov Fujiki form)
② Torelli Thm: S determined by $H^2(S,\mathbb{Z})$	② (Markman–Verbitsky) version of Torelli: X "determined" by $H^2(X,\mathbb{Z})$

Only known deformation types

① Hilbert schemes $S^{[n]}$, S K3 surface — $K3^{[n]}$-type — Beauville

Ex: $M_\sigma(S)$

② $K_n(A)$ generalized Kummer varieties, A abelian surface

③ 6 dim'l

④ 10 dim'l

O'Grady

	Fano	HK
Kod. dim	$-\infty$	0
main tool	bir. geom	Lattice theory
# defs types	finite	??
kinds of exs:	Lots (esp. index small)	4

$\boxed{\text{Starting Observation:}}$

[illegible]

Fano variety $\rightsquigarrow$ HK of K3 type

$\boxed{\text{Example:}}$ (1) Classical example

$Y \subset \mathbb{P}^5$ sm. cubic 4-fold

$F(Y)$ = Fano variety of lines on Y

(Beauville-Donagi) HK of $K3^{[2]}$-type

(2) Lots of recent examples

- Debarre-Voisin '10 $\quad Y \subset Gr(3,10)$
- Iliev-Manivel '16
- Debarre-Kuznetsov '18
- Fatighenti-Mongardi '19

$\boxed{\text{Today:}}$ Reverse this?

HK of $K3^{[n]}$-type $\rightsquigarrow$ geometrically "natural" Fano

$\boxed{\text{Recall:}}$ For L lattice, $(\,,\,)$ and $v \in L$

$\mathrm{div}(v)$ = generator of $(v, L) \subset \mathbb{Z}$

Note: $\mathrm{div}(v) \mid (v,v)$

let X = HK of $K3^{[n]}$-type

λ = ample class on X st $q_X(\lambda) = 2$ $\quad$ ← BBF form

(ie $\mathrm{div}(\lambda) = 1$ or 2)

By Global Torelli: $\exists!$ $\tau_\lambda \in \mathrm{Aut}(X)$ st

$\boxed{H^2(\tau_\lambda)^+ = \mathbb{Z}\lambda}$ $\quad w \in H^{2,0}(X)$

On $H^2(X, \mathbb{Z})$ $\quad$ Reflection b/c $q_X(\lambda) = 2$

$\tau_\lambda^* : \alpha \longmapsto -\alpha + q_X(\lambda, \alpha)\,\lambda$

$\lambda \rightsquigarrow$ involution of X

τ_λ antisymplectic (ie $\tau_\lambda^* \omega = -\omega$)

(Beauville) τ is an antisymplectic involution of HK X

$$\mathrm{Fix}\,\tau(X) = \bigsqcup_i F_i \quad \leftarrow \text{sm Lagrangian}$$

ie $\dim F_i = \frac{1}{2}\dim X$

$\omega|_{F_i} = 0$

$\boxed{\text{Thm:}}$ (FMOS) If (X,λ) pol. HK of $K3^{[n]}$-type st $q_X(\lambda) = 2$. Then

1. \# conn. comp. of $\mathrm{Fix}(\tau_\lambda) = \mathrm{div}(\lambda)$
2. If $\mathrm{div}(\lambda) = 2$, one comp of $\mathrm{Fix}(\tau_\lambda)$ is Fano st $\rho = 1$, index $= 3$

$\boxed{\text{Remarks:}}$ 1. λ w/ above properties exists $\begin{cases} \forall n \text{ if } \mathrm{div}(\lambda) = 1 \\ \Leftrightarrow n \equiv 0 \bmod 4 \text{ if } \mathrm{div}(\lambda) = 2 \end{cases}$

2. why $q_X(\lambda) = 2$? HK $X \circlearrowleft \tau_\lambda$ vary in families of max dimension
(ie codim 1 in mod. space of pol. HK)

3. $\boxed{\text{Conjecture:}}$ "Other comp" in $\mathrm{Fix}(\tau)$ is general type (For $n \leq 4$ ✓)

4. Fanos that appear for $n > 4$ are "new"

$\boxed{\text{Proof Idea:}}$

Numerical data $(X,\lambda) \overset{\text{(markman)}}{\Longrightarrow}$ single deformation class

$\Rightarrow$ Deformation of $\mathrm{Fix}_X(\tau_\lambda)$ independent of (X,λ)

$\Rightarrow$ Just need to prove on a single (X,λ)

$\boxed{\text{Strategy}}$ Degenerate to a singular special fiber

(X_0, τ_0) where $\mathrm{Fix}_{X_0}(\tau_0)$ easy to describe

$X \circlearrowleft \tau$ e.g. $\tau_0 \circlearrowright X_0 = \overline{M}_S(v)$ where

$\downarrow$ $\quad \pi : S \xrightarrow{2:1} \mathbb{P}^2$, τ_0 induced by π

Δ $\quad \Gamma$ ram. curve

$\boxed{\text{Ex:}}$ $n=4$, $X_0 \sim \overline{\mathrm{Hilb}^4(S)}$

$$\mathrm{Fix}_{X_0}(\tau_0) = \underset{\substack{\text{gen. type}}}{\begin{matrix} Y_1 \\ \wr \\ \mathrm{Sym}^4 \Gamma \end{matrix}} \cup \underset{\text{Fano}}{\begin{matrix} Y_2 \\ \wr \\ \mathrm{Sym}^2 \mathbb{P}^2 \end{matrix}}$$

$Y \subset \mathrm{Fix}_{X_0}(\tau_0)$ Fano, Gorenstein, sings in codim ≥ 2, lci outside of codim ≥ 3

(Kollár–Mori) Fano on sing. special fiber extends to Fano on general fiber $\blacksquare$

Final Remarks: Speculation

① Relationship between Fano Y & HK X should be categorical

$\boxed{n=4:}$ Lehn–Lehn–Sorger–van Straten '16

$Y \subset \mathbb{P}^5$ cubic fourfold (Fano)

$\rightsquigarrow X(Y) =$ par. twisted cubics on Y

$\uparrow$ $K3^{[4]}$-type

$\mathrm{Fix}_{X(Y)}(\tau_\lambda) = Y \sqcup W$

$\uparrow$ Fano

(Kuznetsov) $D^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$

K3 category

Bayer - Lahoz - Macri - Stellari; Li - Pertusi - Zhao

Reconstruct $(X(Y), \lambda)$ as moduli space on $\mathcal{A}_Y$

$\boxed{\text{Conjecturally}}$ In general (X, λ) with $\mathrm{div}(\lambda) = 2$

Y Fano

(A) $\exists$ K3 cat. $\mathcal{A} \hookrightarrow D^b(Y)$

(B) Reconstruct (X, λ) as moduli space on $\mathcal{A}$