

Fano varieties associated to hyperkähler manifolds

joint w/ E. Macri, K. O'Grady, G. Sacchà

I. Fano varieties

X Fano if $-K_X$ ample

Ex: \mathbb{P}^n , $X_d \subset \mathbb{P}^n$ $d < n+1$, $X_{d_1, \dots, d_k} \subset \mathbb{P}^n$
(Kollár-Miyaoka-Mori) only fin. many def. classes
of sm. Fano varieties of a given dim

\Rightarrow Attempts to classify (Birational geometry)

Ex: $\dim X \leq 3$

Index (X) = largest $r \geq 1$ st $K_X \equiv -rH$ \swarrow ample div on X

Complete class. $P=1$, index $\geq \dim X - 2$

$P > 1$ index $\geq \frac{1}{2}(\dim X + 1)$

Takeaway: Low index \Rightarrow harder to classify

II. Hyperkähler manifolds

Motivation: S K3 surface

\mathbb{C} -kähler surface, $\Pi_1(S) = 0$, $H^0(S, \Omega_S^2) = \mathbb{C}w$

Option 1: X dim n , $H^0(X, \Omega_X^n) = \mathbb{C}w$ Calabi-Yau

Option 2: X dim, \mathbb{C} -kähler, $\Pi_1(X) = 0$, $H^0(X, \Omega_X^2) = \mathbb{C}w$

Hyperkähler manifold

$\Rightarrow K_X \equiv 0 \Rightarrow X$ is Calabi-Yau

$\&$ $\dim X$ is even

Thm: (Reuville-Roosmolen)

X compact Kähler, $c_1(X) = 0$
 \exists finite étale cover $\Pi: \Pi M_i \rightarrow X$ st each M_i
 one of: \mathbb{C} -tors, CY, HK

Nile Properties

| K3 Surfaces | HK manifolds |
|--|--|
| ① $H^2(S, \mathbb{Z})$ has int.-form \rightarrow lattice theory | ① $H^2(X, \mathbb{Z})$ has integral quad. form (Beauville-Bogomolov Fujiki form) |
| ② Torelli Thm: S determined by $H^2(S, \mathbb{Z})$ | ② (Markman-Verbitsky) version of Torelli: X "determined" by $H^2(X, \mathbb{Z})$ |

only known deformation types

- ① Hilbert schemes $S^{[n]}$, S K3 surface K3^[n]-type
- Beauville
Ex: $M_g(S)$
- ② $K_n(A)$ generalized Kummer varieties, A abelian surface
- ③ $6 \dim' Q$, O'Grady
- ④ $10 \dim' Q$

| | Fano | HK |
|---------------|--------------------------|----------------|
| cod. dim | $-\infty$ | 0 |
| main tool | bir. geom | Lattice theory |
| # defs types | finite | ?? |
| kinds of exs: | Lots (resp. index small) | 4 |

Starting Observation:

Fano variety \rightsquigarrow HK of F3 type

Example: ① Classical example

$Y \subset \mathbb{P}^5$ sm. cubic 4-fold

$F(Y)$ = Fano variety of lines on Y
(Beauville-Donagi) HK of $K3^{[2]}$ -type

② Lots of recent examples

- Debarre-Voisin '10 $Y \subset G(3, 10)$
- Iliev-Manivel '16
- Debarre-Kuznetsov '18
- Fatighenti-Mongardi '19

Today: Reverse this?

HK of $K3^{[n]}$ -type \rightsquigarrow geometrically "natural" Fano

Recall: For L lattice, $(,)$ and $v \in L$

$\text{div}(v)$ = generator of $(v, L) \subset \mathbb{Z}$

Note: $\text{div}(v) \mid (v, v)$

Let X = HK of $K3^{[n]}$ -type \swarrow BBF form

λ = ample class on X st $q_X(\lambda) = 2$

(ie $\text{div}(\lambda) = 1$ or 2)

By Global Torelli: $\exists!$ $\tau_\lambda \in \text{Aut}(X)$ st

$$\boxed{H^2(\tau_\lambda)_+ = \mathbb{Z}\lambda}$$

$w \in H^{2,0}(X)$

Reflection b/c

\swarrow $q_X(\lambda) = 2$

On $H^2(X, \mathbb{Z})$

$$\tau_\lambda^* : \alpha \mapsto -\alpha + q_X(\lambda, \alpha)\lambda$$

$\lambda \rightsquigarrow$ involution of X

τ_λ antisymplectic (ie $\tau_\lambda^* \omega = -\omega$)

(Beauville) τ is an antisymplectic involution of $\mathbb{H}K X$

$$\text{Fix } \tau(X) = \bigsqcup_i F_i \begin{matrix} \leftarrow \text{sm Lagrangian} \\ \text{ie } \dim F_i = \frac{1}{2} \dim X \\ \omega|_{F_i} = 0 \end{matrix}$$

Thm: (FMOS) If (X, λ) pol. $\mathbb{H}K$ of $\mathbb{H}3^{(n)}$ -type st $q_X(\lambda) = 2$. Then

① # conn. comp. of $\text{Fix}(\tau_\lambda) = \text{div}(\lambda)$

② If $\text{div}(\lambda) = 2$, one comp of $\text{Fix}(\tau_\lambda)$ is Fano st $p=1$, index = 3

Remarks: ① λ w/ above properties exists $\begin{cases} \forall n \text{ if } \text{div}(\lambda) = 1 \\ \Leftrightarrow n \equiv 0 \pmod{4} \text{ if } \text{div}(\lambda) = 2 \end{cases}$

② why $q_X(\lambda) = 2$? $\mathbb{H}K X$ vary in families of max dimension $\begin{matrix} \uparrow \\ \tau_\lambda \end{matrix}$ (ie codim 1 in mod. space of sol. $\mathbb{H}K$)

③ **Conjecture:** "Other comp" in $\text{Fix}(\tau)$ is general type (for $n \leq 4$ \checkmark)

④ Fanos that appear for $n > 4$ are "new"

Proof idea:

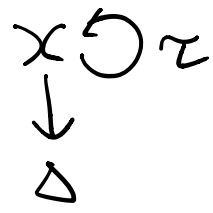
Numerical data (X, λ) $\xRightarrow{\text{(markman)}}$ single deformation class

\Rightarrow Deformation of $\text{Fix}_X(\tau_\lambda)$ independent of (X, λ)

\Rightarrow Just need to prove on a single (X, λ)

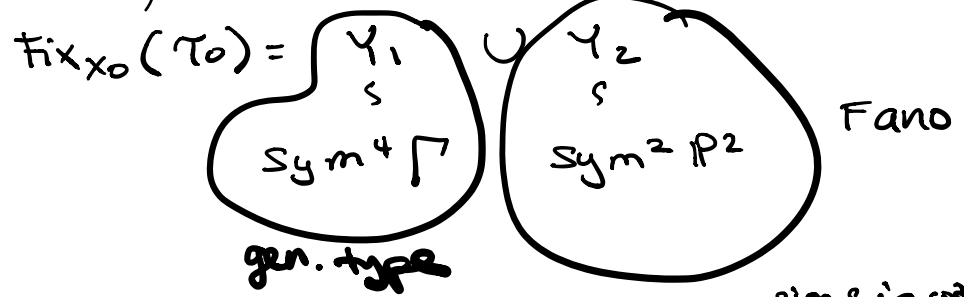
Strategy Degenerate to a singular-special fiber

(X_0, τ_0) where $\text{Fix}_{X_0}(\tau_0)$ easy to describe



e.g. $\tau_0: \mathbb{G}_m \times \mathbb{G}_m = \overline{M}_S(v)$ where
 $\pi: S \xrightarrow{2:1} \mathbb{P}^2$, τ_0 induced by π
 Fram. curve

Ex: $n=4$, $X_0 \sim \overline{\text{Hilb}}^4(S)$



$Y \subset \text{Fix}_{X_0}(\tau_0)$ Fano, Borestein, sing s in codim ≥ 2
 loc outside of codim ≥ 3

(Kollar-Mori) Fano on sing. special fiber extends to Fano on general fiber

Final Remarks: Speculation

① Relationship between Fano Y & HK X should be categorical

n=4: Lehn-Lehn-Sorger-van Straten '16

$Y \subset \mathbb{P}^5$ cubic fourfold (Fano)

$\leadsto X(Y) = \text{par. twisted cubics on } Y$
 \uparrow $K3^{(4)}$ -type

$\text{Fix}_{X(Y)}(\tau_2) = \underbrace{Y}_{\uparrow \text{Fano}} \sqcup W$

(Kuznetsov) $D^b(Y) = \langle \textcircled{A_Y} \otimes_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$
 \uparrow k -category

Bayer - Lahoz - Macrì - Stellari, Li - Pertuisi - Zhao
 Reconstruct $(X(Y), \lambda)$ as moduli space on
 A_Y

Conjecturally In general (X, λ) with $\dim(X) = 2$
 Y Fano

Ⓐ $\cong k$ -cat. $A \hookrightarrow D^b(Y)$

Ⓑ Reconstruct (X, λ) as moduli space on A