λ_g in $CH^*(\overline{\mathcal{M}}_g)$

and in $\log CH^*(\overline{\mathcal{M}}_g)$

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joint work with S. Molcho,
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Notes include comments/corrections after the lecture

Deligne-Mumford moduli spaces of stable curves: $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_{g,n}$

The Hodge bundle:

dualizing sheaf

universal curve
$$\begin{array}{c}\mathcal{C}\\ \pi\downarrow\\ \overline{\mathcal{M}}_g\end{array} \qquad\qquad \mathbb{E}_g = \pi_* \omega_\pi$$

rank g vector bundle on $\overline{\mathcal{M}}_g$

Chern classes:

$$\lambda_i = c_i(\mathbb{E}_g) \in RH^i(\overline{\mathcal{M}}_g) \subset H^{2i}(\overline{\mathcal{M}}_g)$$

$$\lambda_i = c_i(\mathbb{E}_g) \in R^i(\overline{\mathcal{M}}_g) \subset CH^i(\overline{\mathcal{M}}_g)$$

The top Chern class

$$\lambda_g = c_{top}(\mathbb{E}_g)$$

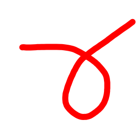
plays an important role:

- Vanishing properties

$$\lambda_g^2 = 0 \quad \text{on} \quad \overline{M}_g$$

$$\lambda_g \big|_{\Delta_0} = 0 \quad \text{on} \quad \Delta_0$$

$\Delta_0 \subset \overline{M}_g$ divisor of curves



with non-separating node

Mumford's Identity 1983

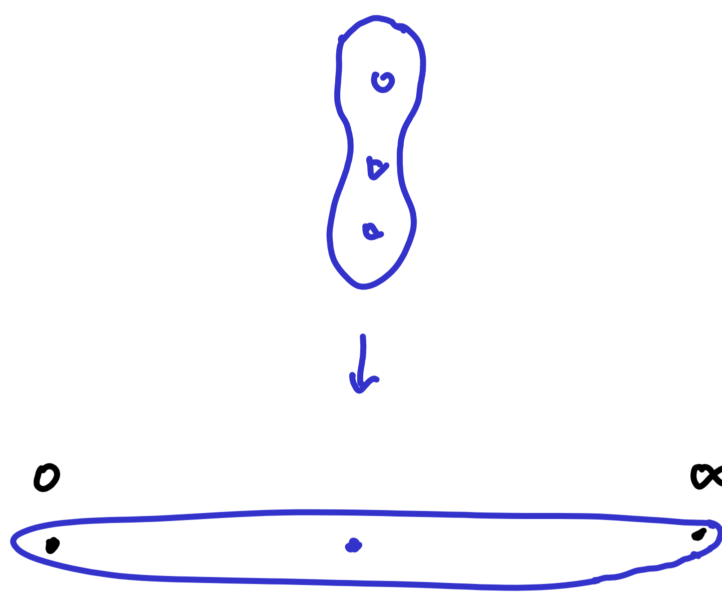
$$c(\mathbb{E}_g)\cdot c(\mathbb{E}_g^\vee) = 1$$

Trivial quotient $\mathbb{C}$

obtained from residue at the node

- $(-1)^g \lambda_g$ appears in the virtual fundamental class of moduli of contracted maps in Gromov-Witten theory for target curves:

$$DR_{g,(0,\dots,0)} = (-1)^g \lambda_g$$



Consequence of connection to GW theory: λ_g formula

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \ldots, k_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g$$

$\psi_i = c_1(\mathbb{L}_i)$

i^{th} cotangent line

Virasoro constraints [Getzler - P, 1998]

Localization formulas [Faber - P, 2000]

Kappa rings of $M_{g,n}^{ct}$ [P, 2012]

λ_g Surface theories [Bousseau 2018]

Katz-Klemm-Vafa formula, Quantum tropical vertex

- $(-1)^g \lambda_g$ arises as the pull-back of the universal 0-section of the moduli space of PPAVs.

Principally Polarized Abelian Varieties

Hain 2013

Grushevsky - Zakharov 2014

Goals here:

(i) Bound from below the complexity of

$$\lambda_g \in CH^*(\bar{M}_g)$$

(ii) Show $\lambda_g \in \log CH^*(\bar{M}_g)$ is (almost) as simple as possible.

Motivations :

Starting point is a beautiful formula over A_g ← Moduli of PPAVs of dim g

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \\ A_g \arrow[u, bend right, "s"'] \end{tikzcd}$$

Universal abelian variety

0-section

Let $Z_g \in CH^g(\mathcal{X}_g)$ be the class of the 0-section. Then:

$$\boxed{Z_g = \frac{\Theta^g}{g!} \in CH^g(\mathcal{X})}$$

via FM by Deninger, Murre

where $\Theta \in CH^1(\mathcal{X})$ is the universal symmetric theta divisor trivialized along s

We can pull-back the formula via the Torelli map:

$A = (a_1, \dots, a_n)$

$\sum a_i = 0$

$$
\begin{tikzcd}
\mathrm{Jac} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathcal{M}_{g,n}^{ct} \arrow[r, "\tau"] \arrow[u, bend left=70, "AJ"] \arrow[u, bend left=25, "0"'] & \mathcal{A}_g
\end{tikzcd}
$$

We conclude (Grushevsky - Zakharov, 2014)

$$
DR_{g,A} = \frac{T^g}{g!} \in CH^g(\mathcal{M}_{g,n}^{ct})
$$

for $T \in CH^1(\mathcal{M}_{g,n}^{ct})$

Comuputed by Grushevsky - Zakharov

Question: Could such a formula hold over $\overline{\mathcal{M}}_g$?

Simplest case: Could we have

$$(-1)^g \lambda_g = \frac{T^g}{g!} \in CH^g(\overline{\mathcal{M}}_g)$$

for some $T \in CH^1(\overline{\mathcal{M}}_g)$

with $T|_{\mathcal{M}_g^{ct}} = 0$?

GZ calculation for $DR_{g,\phi}$, implies $\lambda_g|_{\mathcal{M}_g^{ct}} = 0$

Answer: No, even for $g=2$.

Maybe we should ask for a bit less.

Let $\operatorname{div} CH^*(\overline{\mathcal{M}}_g) \subset CH^*(\overline{\mathcal{M}}_g)$

be the subalgebra generated by $CH^1(\overline{\mathcal{M}}_g)$.

Question: Is $\lambda_g \in \operatorname{div} CH^*(\overline{\mathcal{M}}_g)$?

Theorem 1: For all $g \geq 3$,

$$\lambda_g \notin \operatorname{div} CH^*(\overline{\mathcal{M}}_g)$$

Proof: $\lambda_2 = \frac{\lambda_1^2}{2} \in \operatorname{div} CH^*(\overline{\mathcal{M}}_2)$

$\lambda_3 \notin \operatorname{div} CH^*(\overline{\mathcal{M}}_3)$

$\lambda_3 \in \operatorname{div} CH^*(\overline{\mathcal{M}}_{3,1})$ ← so far unexplained

admcycles: We know $R^*(\overline{\mathcal{M}}_3)$ and $R^*(\overline{\mathcal{M}}_{3,1})$ completely.

$$\operatorname{div} R^3(\overline{M}_3) \subset R^3(\overline{M}_3)$$

rank 9 rank 10

$$\rotatebox{90}{$\notin$}$$

$$\lambda_3$$

Surprising

$$\operatorname{div} R^3(\overline{M}_{3,1}) \subset R^3(\overline{M}_{3,1})$$

rank 28 rank 29

$$\rotatebox{90}{$\in$}$$

$$\lambda_3$$

$$\operatorname{div} R^4(\overline{M}_{4,1}) \subset R^4(\overline{M}_{4,1})$$

rank 103 rank 191

$$\rotatebox{90}{$\notin$}$$

$$\lambda_4$$

so $\lambda_4 \notin \operatorname{div} R^4(\overline{M}_4)$

Requires admcycles in $g=4$

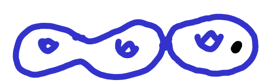
For $g \geq 5$, we use a boundary argument

Suppose (for contradiction) $\lambda_g \in \operatorname{div} R^g(\overline{\mathcal{M}}_g)$,

Then $\lambda_g \in \operatorname{div} R^g(\overline{\mathcal{M}}_{g,1})$.

Boundary map

$$\delta : \quad \overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{g,1}$$



We find

(i) $\quad \delta^* \lambda_g = \lambda_{g-1} \otimes \lambda_1$

(ii) $\quad \delta^* \lambda_g \in \operatorname{div} CH^*(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2})$

Affine stratification

so $\quad \delta^* \lambda_g \in \operatorname{div} R^*(\bar{\mathcal{M}}_{g-1,1}) \otimes \operatorname{div} R^*(\bar{\mathcal{M}}_{1,2})$

using (i) we have

$$\pi_{1*}(\delta^* \lambda_g \cdot \psi_1) = \frac{1}{24} \lambda_{g-1} \in R^{g-1}(\bar{\mathcal{M}}_{g-1,1})$$

projection to $\bar{\mathcal{M}}_{g-1,1}$

cotangent line on $\bar{\mathcal{M}}_{1,2}$

using (ii) we have

$$\pi_{1*}(\delta^* \lambda_g \cdot \psi_1) \in \operatorname{div} R^*(\bar{\mathcal{M}}_{g-1,1})$$

We have proven:

$$\boxed{\lambda_g \in \operatorname{div} R^g(\overline{\mathcal{M}}_{g,1}) \Rightarrow \lambda_{g-1} \in \operatorname{div} R^{g-1}(\overline{\mathcal{M}}_{g-1,1})}$$

Since we know $\lambda_4 \notin \operatorname{div} R^4(\overline{\mathcal{M}}_{g-1,1})$

We conclude $\lambda_g \notin \operatorname{div} R^g(\overline{\mathcal{M}}_{g,1})$

and thus $\lambda_g \notin \operatorname{div} R^g(\overline{\mathcal{M}}_g)$

for $g \geq 5$ $\square$

genus 8 is needed for CH^* results

<u>Theorem 2:</u> For all $g \geq 8$,

λ_g is not in the subalgebra of $CH^*(\overline{\mathcal{M}}_g)$ generated by $CH^{\leq 2}(\overline{\mathcal{M}}_g)$

Requires a much more difficult initial case:

rank 1314 in degree 5

λ_5 is not in the subalgebra of $R^5(\bar{M}_{5,1})$ generated by $R^{\leq 2}(\bar{M}_{5,1})$

Faber points out should use R^* here

rank 1371

- admcycles
- 9665 decorated strata
- 31 days / MPI computer 60 G of RAM

Conjecture: for fixed k,

λ_g is in the subalgebra of $CH^*(\bar{M}_g)$ generated by $CH^{\leq k}(\bar{M}_g)$ for only finitely many g.

$$\boxed{\text{Conclusion: } \lambda_g \text{ is as complex as possible}}$$

Two remarks:

I admcycles proves Pixton's Conjecture about relations in new cases

$$R^4(\bar{\mathcal{M}}_{4,1}), \qquad R^5(\bar{\mathcal{M}}_{5,1})$$

II Torelli $\quad \bar{\mathcal{M}}_g \to \bar{A}_g \quad$ Alekseev, 2nd Voronoi, Olsson

Theorems 1 + 2 bound from below the complexity of

$$\bar{Z}_g \in CH^g_{op}(\bar{\mathcal{X}}_g)$$

universal
$\bar{\mathcal{X}}$
$\downarrow$
$\bar{A}_g$

Divisor bound can also be obtained by geometric study by Grushevsky-Zakharov 2014

What is $\log CH^*(\overline{M}_g)$?

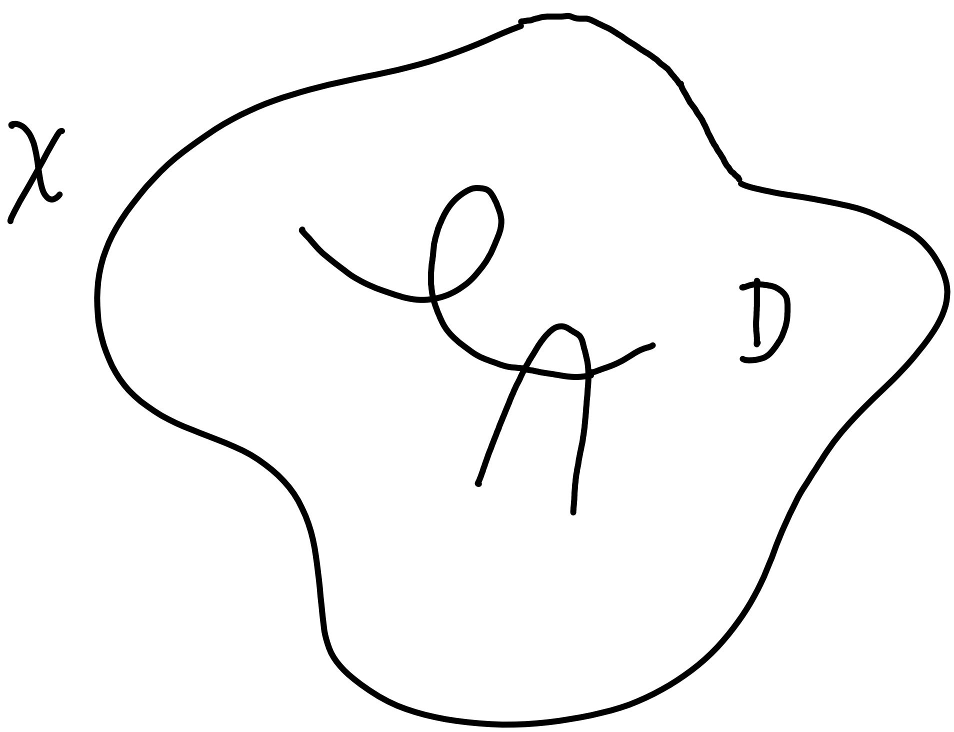
Suppose we have a pair

$$(X, D)$$

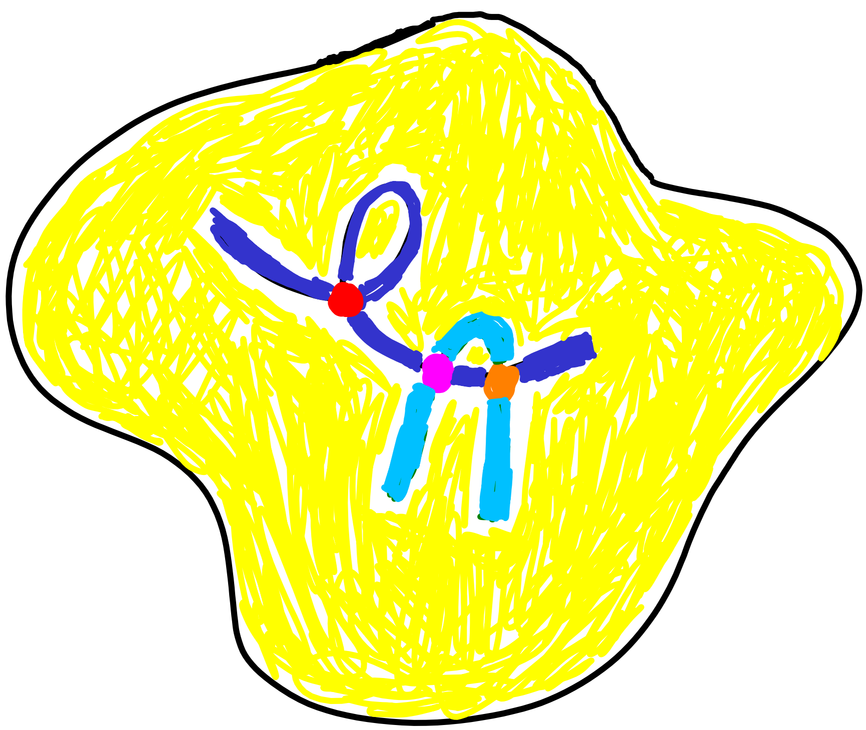
nonsingular variety (or DM stack)

Normal crossings divisor

Not assumed strict normal crossings



There is a basic notion of Strata for (X, D)



Strata indicated by colors

A stratum $S \subset X$ is quasiprojective and nonsingular. The closure

$$\bar{S} \subset X$$

may be singular.

A simple blow-up of (X, D) is a blow-up along a nonsingular stratum closure $\bar{S} \subsetneq X$

$$Bl : (\hat{X}, \hat{D}) \to (X, D)$$

Strict transform of D union the exceptional divisor E

We define a Category

$$\mathcal{B}(X, D)$$

- Objects of $\mathcal{B}(X,D)$ are

$$(\hat{X},\hat{D}) \xrightarrow{\hat{\phi}} (X,D)$$

where $\hat{\phi}$ is a composition of simple blow-ups.

- Morphisms are diagrams :

γ is a composition of simple blow-ups

$$\begin{tikzcd}
(\hat{\hat{X}},\hat{\hat{D}}) \arrow[rr, "\gamma"] \arrow[dr, "\hat{\hat{\phi}}"'] & \arrow[d, phantom, "\circlearrowright"] & (\hat{X},\hat{D}) \arrow[dl, "\hat{\phi}"] \\
& (X,D) &
\end{tikzcd}$$

- $\log CH^*(X, D)$

$$\| \text{ definition}$$

$$\lim_{\substack{\longrightarrow \\ (\hat{X}, \hat{D}) \in \mathcal{B}(X, D)}} CH^*(\hat{X})$$

- $\text{div} \log CH^*(X, D) \subset \log CH^*(X, D)$

Subalgebra generated by boundary divisors

(strictest definition)

Wise view: $\boxed{\text{The combinatorial part}}$

$$(\overline{\mathcal{M}}_g, \partial \overline{\mathcal{M}}_g)$$ ← nonsingular with normal crossings

Theorem 3. For all g,

$$\lambda_g \in \operatorname{div} \log CH^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g).$$

$$\boxed{\text{From the log perspective, } \lambda_g \text{ is as simple as possible, an almost combinatorial object.}}$$

Question by Zakharov: Is the parallel statement true for $\overline{\mathcal{Z}}_g \subset \overline{\mathcal{X}}_g$? Answer: Don't know.

The proof has two main Steps:

I. General result about normally decorated strata classes.

II. Pixton's formula for λ_g.

Start with Step I

$$S \subset X \quad , \quad (X, D)$$

quasi proj stratum

Normal crossings

$$\overline{S} \subset X$$

nonsingular variety (or DM stack)

Closure

$\bar{S} \subset X$ may be singular,

$$\tilde{S} \to \bar{S} \subset X$$

Normalization

Then $\tilde{S}$ is nonsingular

and carries a normal bundle

$Nor(\tilde{S}, X)$ since the map

$\tilde{S} \xrightarrow{i} X$ is an immersion.

Moreover: $Nor(\tilde{S}, X) = \bigoplus_j Nor_j(\tilde{S}, X)$

The splitting

$$Nor(\tilde{S}, x) = \bigoplus_j Nor_j(\tilde{S}, x)$$

corresponds to the monodromy orbits of the branches of D which define S.

An example of a normally decorated strata class is

$$i: \tilde{S} \to X \qquad i_* \left(\prod_j P_j \right) \in CH^*(X)$$

where P_j is a polynomial in $c_k(Nor_j(\tilde{S}, x))$

The most general normally decorated strata class is defined as follows.

Let G be the monodromy action on the branches of D which define S.

Let P be any G-invariant polynomial in the chern roots of $Nor(\tilde{S}, X)$ defined on a G-cover of $\tilde{S}$ associated the branches. Then,

$$\underbrace{i_*}_{i:\tilde{S}\to X}(P) \in CH^*(X)$$

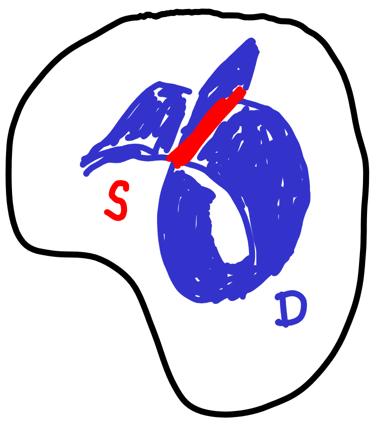
is a normally decorated strata class

Theorem 4: Every normally decorated strata class lies in $\mathrm{divlogCH}^*(X, D)$.

Claim is trivial when $\bar{S} \subset X$ is cut out transversally by distinct components $D_1, \dots, D_{\mathrm{codim}\, S}$.

The case with self intersections is somewhat painful to express in words ... [Warning: proof to be written]

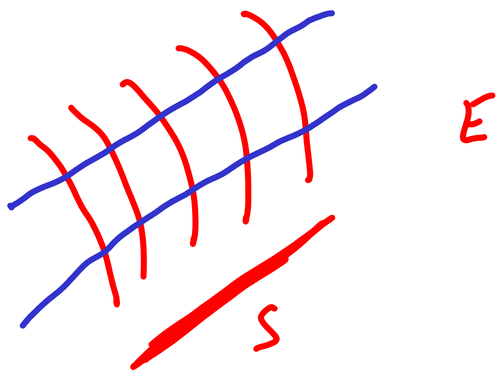
Simple Example:



Claim: $[S] \in \operatorname{div} \log CH^*$

Could have monodromy

Blow up S



Then $\pi^*[S] = (2E + \hat{D})E - E^2$

Strict transform of D

II Pixton's formula for $DR_{g,A}$

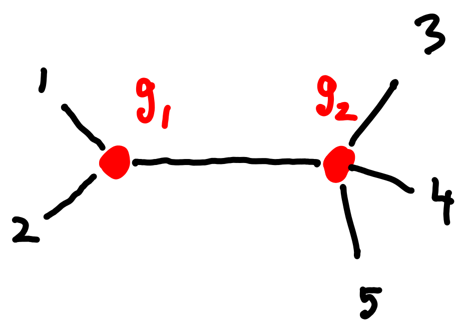
Let $G_{g,n}$ = set of stable graphs of genus g with n markings

finite set

for $\Gamma \in G_{g,n} \rightsquigarrow \overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$

product of moduli spaces determinded by the vertices

for Γ =



$$\overline{M}_\Gamma = \overline{M}_{g_1,3} \times \overline{M}_{g_2,4}$$

Tautological classes are given by

$$\xi_{\Gamma *}\Big(\underbrace{\prod \psi_i^{m_i}}_{\text{markings}} \ \underbrace{\prod \psi_j^{n_j}}_{\text{halves of edges}} \ \underbrace{\prod \text{kappas}}_{\text{vertices}} \Big) \in R^*(\overline{\mathcal{M}}_{g,n})$$

$$\overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$$

The linear span of all such classes defines the tautological ring

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset CH^*(\overline{\mathcal{M}}_{g,n}).$$

[Faber-P : Tautological and Non-Tautoligcal classes]

[P : A calculus for the moduli of curves]

Let $\Gamma \in G_{g,n}$ be a stable graph.

Let r be a positive integer

A weighting mod r of Γ is

$$w : \mathcal{H}(\Gamma) \to \{0, 1, \dots, r-1\}$$

half edges

Remember

$A = (a_1, \dots, a_n)$

$\sum a_i = 0$

(I) $i \in$ Marking, $w(i) = a_i \mod r$

(II) $e = (h, h') \in$ Edge, $w(h) + w(h') = 0 \mod r$

(III) $v \in$ Vertex, $\sum_{h \vdash v} w(h) = 0 \mod r$

$W_{\Gamma, r}$ is set of weightings mod r of Γ

$|W_{\Gamma,r}| = r^{h^1(\Gamma)}$

Definition (Pixton)

Let $P_g^{d,r}(A) \in R^d(\overline{M}_{g,n})$ be the degree d component of

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\left[\prod_{i=1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i\right) \cdot \prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}} \right]$$

Various motivations: Compact type restriction,

Givental - Teleman theory

Claim (Pixton):

Ehrhart Theory, see JPPZ

$P_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ is polynomial in r for all $r \gg 0$.

Definition (Pixton):

$P_g^d(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ is the constant term of $P_g^{d,r}(A)$

$r=0$

Theorem (Conjectured by Pixton, proven in JPPZ 2016)

$$DR_{g,A} = P_g^g(A) \in R^g(\overline{\mathcal{M}}_{g,n}).$$

Formula for $\lambda_g = c_g(\mathbb{E}_g)$

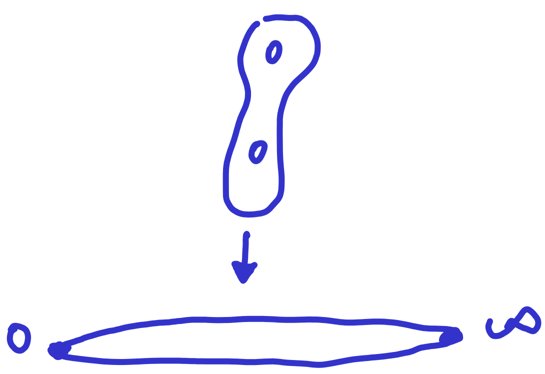
supported on $\Delta_0 \subset \overline{\mathcal{M}}_g$

We view $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_{g,0}$

$n=0$

Let $A = \emptyset$

Geometry $\Rightarrow$ $\overline{\mathcal{M}}_g(\mathbb{P}^1, \emptyset)^{\sim} \cong \overline{\mathcal{M}}_g$



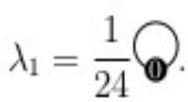
Moreover $[\overline{\mathcal{M}}_g(\mathbb{P}^1, \emptyset)^{\sim}]^{vir} = (-1)^g \lambda_g$

$$DR_g(\emptyset) = (-1)^g \lambda_g$$

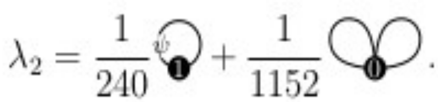
So we can apply the DR cycle Formula:

Diagrams from JPPZ

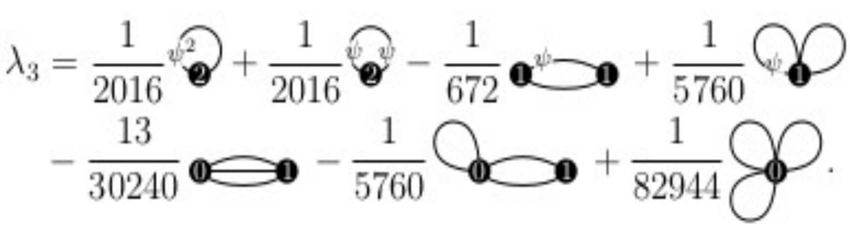
Genus 1.



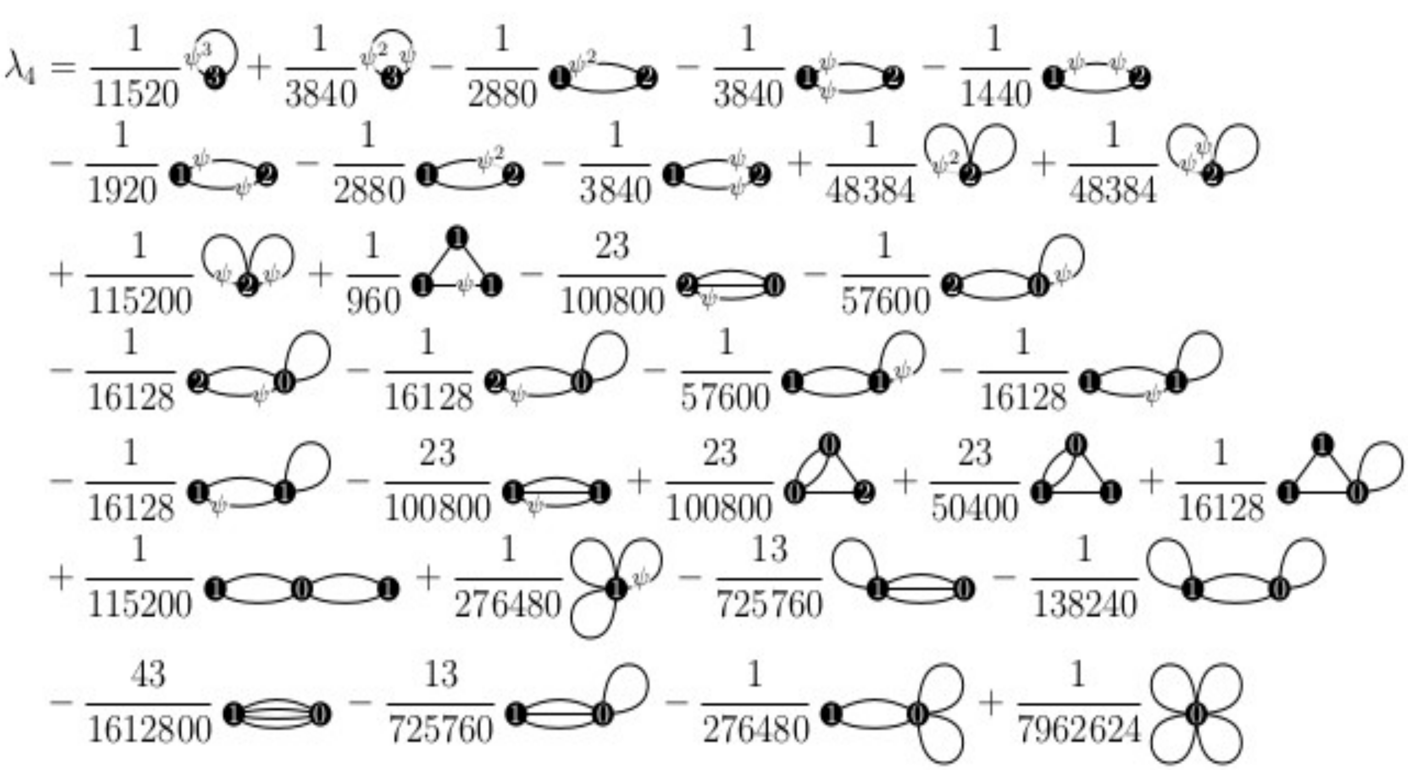
Genus 2.



Genus 3.



Genus 4.



It is a remarkable formula:

- λ_g is a sum of normally decorated strata classes

- all the strata which appear are in Δ_0

The proof of Theorem 4 is complete. $\square$

Question by Faber, Vakil:

$\lambda_g \in \mathrm{divlog}(\overline{M}_g, \Delta_0)$?

Answer: probably yes.

Similar argument shows

$$DR_{g,A} \in \underline{\mathrm{div}} \log CH^*(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$$

not only boundary divisors but also ψ_i's

There is a finer class

$$DR^{\log}_{g,A} \in \log CH^*(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$$

which is not quite equal

$$DR^{\log}_{g,A} \neq DR_{g,A}$$

There is an ongoing study of the difference (from various perspectives)

T. Graber

D. Holmes

D. Ranganathan

J. Wise

and us SM, RP, JS

Conjecture (very likely!):

$$D^{\log}_{g,A} \in \underline{\mathrm{div}} \log CH^*(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$$

The End