# Virtual Cycle on the Moduli of Maps to a Complete Intersection

w/ Qile Chen, Felix Janda

$Y$ smooth proj variety

$E$ vector bundle

$s : Y \to E$ regular section

$X = V(s)$

$$\begin{tikzcd} & E \arrow[d] \\ X \subset & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

Thm (Fulton IT)

There is a localized Euler class $e_s(E) \in A_*(X)$

st $\iota_* e_s(E) = e(E) \cap [Y]$ in $A_*(Y)$ $\qquad e_s(E) = [X]$ in $A_*(X)$

Thm (Chen-Janda-W)

There is a moduli space of p-fields $\mathcal{M} = \mathcal{M}(Y,E)$ w/ canonical P.O.T AND cosection $\sigma$ whose "vanishing" locus is $\bar{\mathcal{M}}(X) \subset \mathcal{M}$.


There is a cosection localized V.F.C.

$[\mathcal{M}]^{vir}_\sigma \in A_*(\bar{\mathcal{M}}(X))$ $\quad$ st $\quad [\mathcal{M}]^{vir}_\sigma = (-1)^{\chi(E, \mathcal{M}(X))} [\bar{\mathcal{M}}(X)]^{vir}$

$\iota_* [\mathcal{M}]^{vir}_\sigma = [\mathcal{M}]^{vir}$

$C \xrightarrow{f} X \qquad \chi(f^* E)$

## A Gromov-Witten Problem

Def Gromov Witten invariants of $Y$ are numbers defined as follows:

$$\begin{tikzcd} \bar{\mathcal{M}}(Y) := \bar{\mathcal{M}}_{g,n}(Y,\beta)(S) = \left\{ C \to Y,\ C \downarrow S \text{ stable map} \right\} \arrow[d] \arrow[r, "ev_i"] & Y \\ \mathcal{M}_{g,n} & \end{tikzcd}$$

Key: $\bar{\mathcal{M}}(Y)$ has a canonical relative perfect obstruction theory $\Rightarrow [\bar{\mathcal{M}}(Y)]^{vir} \in A_*(\bar{\mathcal{M}}(Y)) \Rightarrow \int_{[\bar{\mathcal{M}}(Y)]^{vir}} ev_i^* \alpha_i$, $\quad \alpha_i \in H^*(Y)$

Problem "compute" the GW invariants of $Q \subset \mathbb{P}^4$ $\qquad \mathcal{O}(5) \downarrow \mathbb{P}^4$, section $\sum x_i^5$

- genus 0: Givental, Lian-Liu-Yau 90's
  - Quantum Lefschetz
  - Torus localization
- genus 1: Zinger, Vakil .... mid 00's
- genus > 1?

(one) (partial) solution moduli of p-fields

- moduli due Chang-Li (quintic)
- thm for virtual classes by Chang-Li, Kim-Oh ($V /\!/ G$)
- used by Chen, Guo, Janda, Ruan to compute $g=2$ invariants

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Kiem-Li

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## Moduli of p-fields

Define $\mathcal{M}(Y,E)(S) = \left\{ \begin{tikzcd} f^*\mathcal{E}^\vee \otimes \omega \arrow[d] & \\ C \arrow[u, bend left, "s"] \arrow[r, "f"] \arrow[d] & Y \\ S & \end{tikzcd} \;\middle|\; \begin{array}{l} f \text{ stable map} \\ s \text{ section} \end{array} \right\}$

## Moduli of sections:

Given $\begin{tikzcd} Z \arrow[d] \\ \mathcal{C} \arrow[d] \\ \mathcal{U} \end{tikzcd}$ define $\mathrm{Sec}_{\mathcal{U}}(Z/\mathcal{C})(S) = \left\{ \begin{tikzcd} & Z \arrow[d] \\ C_S \arrow[ur] \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ S \arrow[r] & \mathcal{U} \end{tikzcd} \right\}$

**Thm** $\mathrm{Sec}_{\mathcal{U}}(Z/\mathcal{C})$ is an algebraic stack w/ an obstruction theory

**Examples** ① $Z = \mathcal{C}_{g,n} \times Y$, $\mathcal{U} = \mathfrak{M}_{g,n}$ $\qquad \mathrm{Sec}_{\mathcal{U}}(Z/\mathcal{C}) = \mathfrak{M}_{g,n}(Y) \supset \overline{\mathcal{M}}(Y)$

② $Z = \mathcal{E}^\vee \otimes \omega$ $\qquad \mathrm{Sec}_{\mathcal{U}}(Z/\mathcal{C}) = \mathfrak{M}_{g,n}(Y,E) \supset \mathcal{M}(Y,E)$

$$\begin{tikzcd} \mathcal{M}(Y,E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{M}_{g,n}(Y,E) \arrow[d] \\ \overline{\mathcal{M}}(Y) \arrow[r] & \mathfrak{M}_{g,n}(Y) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}^\vee \otimes \omega = Z \arrow[d] \\ \mathcal{C}_{g,n} \times Y \arrow[d] \\ \mathcal{C}_{g,n} = \mathcal{C} \arrow[d] \\ \mathfrak{M}_{g,n} = \mathcal{U} \end{tikzcd}$$

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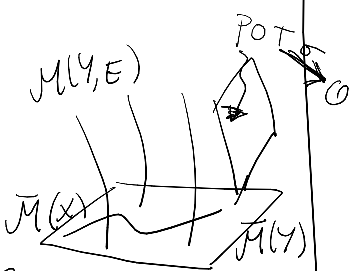


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$C \xrightarrow{f} X$     $\chi(f^*E)$

# Virtual Fundamental Classes

$\mathcal{M} = \mathcal{M}(Y,E)$   $\mathfrak{m} = \mathfrak{m}_{g,n}$

Thm (Behrend-Fantechi) Given $\phi : \mathbb{E} \to L_{\mathcal{M}/\mathfrak{m}}$ a relative P.O.T. then get $[\mathcal{M}]^{vir} \in A_*(\mathcal{M})$

$\quad \| \; 0^!_{h^1/h^0(\mathbb{E}^\vee)} [C_{\mathcal{M}/\mathfrak{m}}]$

Examples ① If $\mathcal{M} \to \bullet$ is a smooth scheme then $(id : L_{\mathcal{M}} \to L_{\mathcal{M}}) = (id : \Omega_{\mathcal{M}} \to \Omega_{\mathcal{M}})$ is a POT and $[\mathcal{M}]^{vir}_\sigma = [\mathcal{M}]$   $\Omega_{\mathcal{M}}^\vee \xrightarrow{0} \mathcal{O}$

② If $E$ is a vector bundle $Y (=\mathcal{M})$ a smooth scheme, then

$$\begin{tikzcd} \mathbb{E} \arrow[d, "\phi"'] & {[E} \arrow[r, "0"] \arrow[d] & {\Omega_Y]} \arrow[d, equal] \\ L_Y & {[0} \arrow[r] & {\Omega_Y]} \\ & \text{deg } -1 & \text{deg } 0 \end{tikzcd}$$

$E^\vee \xrightarrow{W} \mathcal{O}$
$(f,y) \longmapsto f(s(y))$

is a P.O.T.

$[Y]^{vir} = e(E^\vee) \cap [Y]$

$[Y]^{vir}_\sigma = e_s(E^\vee)$

③ Canonical OT on $\overline{\mathcal{M}}(X), \overline{\mathcal{M}}(Y), \mathcal{M}(Y,E)$

$\Rightarrow [\cdot]^{vir}$ for each one

Thm (Kiem-Li) Given $\phi : E \to L_{\mathcal{M}/\mathfrak{m}}$ a relative POT AND $\sigma : E^\vee \to \mathcal{O}_{\mathcal{M}}[1]$ st $\sigma \circ \phi^\vee = 0$

get $[\mathcal{M}]^{vir}_\sigma \in A_*(\mathcal{M}(\sigma))$

"$0^!_{E(\sigma)} [C_{\mathcal{M}/\mathfrak{m}}]$"

locus where $\sigma$ "vanishes"

COSECTION EXAMPLES

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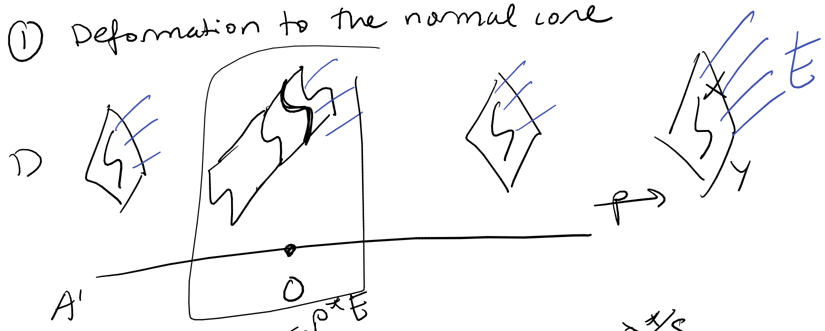
# Proof of Fulton, via our methods

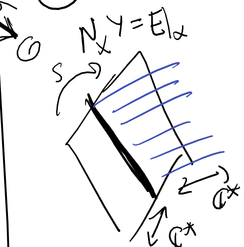
① Deformation to the normal cone




Have $\cancel{E_D} = p^*E$, $E_D(-D_0)$ and a section $\cancel{p^*s}$ $\tilde{s}$

this is a rational equivalence of $e_s(E_D(-D_0))$ in fibers over 0 and not 0.

② Torus localization in special fiber




use this POT $\mathbb{E}^\bullet = [E \xrightarrow{0} T^\vee_{E:E|_X}]$

Thm $[E]^{vir}_{\mathbb{E},W} = \iota_* \dfrac{[F]^{vir}_{\mathbb{E}|_F^{fix},W}}{e_{\mathbb{C}^*}((\mathbb{E}|_F^{mov})^\vee)}$

• Fixed locus is $X \subset N_X Y$ via zero section

$\mathbb{E}|_X^{fix} = [0 \to T_F^\vee] \qquad \mathbb{E}|_X^{mov} = [E \to N^\vee_{X/E}]$, $N^\vee_{X/E} = E^\vee$

$$e_s(E^\vee) = \frac{[X]}{(-1)^{rk(E)}}$$