I. Question.

Def. A Severi-Brauer variety is a sm. proper geo. connected k-scheme X s.t.

$$X_{\bar{k}} \cong \mathbb{P}^{d-1}_{\bar{k}}. \quad \text{degree of } X.$$

$\mathrm{Aut}\, \mathbb{P}^{d-1} \cong PGL_d$

$$1 \to \mathbb{G}_m \to GL_d \to PGL_d \to 1$$

$$\lambda \mapsto \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

$$\{\deg d.\ \text{SB vars}\}/_{\text{iso}} \cong H^1_{et}(\mathrm{Spec}\, k, PGL_d)$$

Ex. $C \subseteq \mathbb{P}^2_k$ cut out $ax^2+by^2=z^2$ defines a SB var. of deg 2.

$$C_{\bar{k}} \cong \mathbb{P}^1_{\bar{k}}.$$

$a, b \in k^*$

$$A \cong k\langle x, y\rangle / x^2=a,\ y^2=b,\ xy=-yx$$

Gen. quaternion algebras.

Ex. A central simple alg. of degree d (twisted form of $M_d(k)$), then

$SB(A)$ = moduli of d-dim left ideals in A.

Clark, Saltman

Q. If X is a SB variety, is there a genus 1 curve C with a map $C \to X$?

Ex. $X \cong \mathbb{P}^{d-1}$, yes.

Reformulation.

$$X \longmapsto \alpha_X = \alpha \quad \text{Brauer class}$$

$$\begin{array}{ccccc} H^1(\mathrm{Spec}\, k, GL_d) & \to & H^1(\mathrm{Spec}\, k, PGL_d) & \longrightarrow & H^2(\mathrm{Spec}\, k, \mathbb{G}_m) \\ \wr\| & & & & \wr\| \\ * & & & & Br(k) \end{array}$$

$d\alpha = 0.$

$$C \longrightarrow X \iff \begin{matrix} C \\ \downarrow \\ \operatorname{Spec} k \end{matrix} \quad \alpha|_C = 0 \ \text{ in } \operatorname{Br}(C) = H^2(C, \mathbb{G}_m)$$

$$\iff \alpha|_{k(C)} = 0 \ \text{ in } \operatorname{Br}(k(C))$$

<u>Rem</u>. $\alpha \neq 0$ and C splits α $(C \to X)$

$\implies C(k) = \phi.$

<u>Rem</u>. $X(k) \neq \phi$, $X \cong \mathbb{P}^{d-1}_k$.

II. Degrees 2–6. Suets, de Jong – Ho, Auel.

<u>Degree 2</u>. X degree 2. Twisted form of $\mathbb{P}^1_k$.

No "$\mathcal{O}(1)$".

There is an "$\mathcal{O}(2)$".

Take two general members D_1, D_2

$D_1 + D_2$ effective divisor of degree 4.

$$C \xrightarrow[\substack{\text{branched at} \\ D_1 + D_2}]{2:1} X$$

Riemann – Hurwicz $\implies g(C) = 1.$

<u>Degree 3</u>. X tw. form of $\mathbb{P}^2_k$.

There is an "$\mathcal{O}(3)$".

General section is a twisted form

of a cubic curve.
Here $g = 1$.

Degree 4. Twist the "intersection of two quadrics" in $\mathbb{P}^3$.

Degree 5. More.

Degree 6 (Avel). Play around with Segre embedding
$$\underline{\mathbb{P}^1 \times \mathbb{P}^2} \hookrightarrow \mathbb{P}^5.$$

$$\begin{array}{ccc} X \times Y & \hookrightarrow & Z \\ \uparrow & & \\ C \times D & & \end{array}$$

Bad news: No complete answer in any higher degree.

III. Why genus 1?

It's the only genus that works.

C has genus g, then it has an effective 0-cycle of deg $2g-2$.

But X typically only has 0-cycles divisible by d.

$C \longrightarrow X$

$d \mid (2g-2).$

Thm (Ho–Lieblich, Ar–Aul). Any SB X is split by a torsor for an abelian variety.

Best you can do for $d = 7$?

235 299.

'Yes' for local fields.

$Br(k) \cong \mathbb{Q}/\mathbb{Z}$

$C \in H(k, E_n)$

$ord(C) = n \Rightarrow C$ splits

$\frac{1}{n} \in \mathbb{Q}/\mathbb{Z}.$

IV. A harder question.

$$H^2(\mathrm{Spec}\, k, \mu_n) \cong H^2(\mathrm{Spec}\, k, \mathbb{G}_m)[n]$$

$$\beta \longmapsto \alpha$$

Q. Is there a g1c that splits β? I.e.,

$$\beta|_C = 0 \text{ in } H^2(C,\mu_n) \xrightarrow[\text{injective}]{\text{not}} H^2(k(C),\mu_n).$$

Provably stronger condition.

$$C_{\bar{k}} \to C$$

$\mathbb{G}_m$

$$\begin{array}{ccc} H^1(C_{\bar{k}},\mathbb{G}_m)^G & \xrightarrow{d_2} & H^2(k,\mathbb{G}_m) \\ \wr\| & & \\ \mathrm{Pic}(C_{\bar{k}})^G & & \end{array}$$

μ_n

$$\begin{array}{ccc} H^1(C_{\bar{k}},\mu_n)^G & \xrightarrow{d_2} & H^2(\mathrm{Spec}\, k,\mu_n) \\ \wr\| & & \\ \mathrm{Pic}(C_{\bar{k}})[n]^G & & \end{array}$$

V. Splitting cyclic algebras / SB varieties.

$$H^1(\mathbb{Z}/n) \times H^1(\mu_n) \to H^2(\mu_n) \to H^2(\mathbb{G}_m)$$

$$(\chi, u) \longmapsto (\chi, u).$$

Cyclic Brauer classes.

Major Open Q. Is every deg. p SB variety cyclic, where p is a prime.

Yes $p = 2, 3$.

E' elliptic curve w/ n-torsion point P

$$\mathbb{Z}/n \subseteq E'$$

$$0 \to \mathbb{Z}/n \to E' \to E \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & \mu_n \arrow[r] & E \arrow[r] & E' \arrow[r] & 0 \\
0 \arrow[r] & \mu_n \arrow[r] \arrow[u, equal] & E[n] \arrow[r] \arrow[u] & \mathbb{Z}/n \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$$0 \to \mu_n(k) \to E[n](k) \to \mathbb{Z}/n \to H^1(\mathrm{Spec}\, k, \mu_n)$$

$$P \mapsto u$$

depends on $E[n]$.

$$x \mapsto (x, u) = x \cup u$$

$$\begin{tikzcd}
H^1(\mathrm{Spec}\, k, \mathbb{Z}/n) \arrow[r] \arrow[d] & H^2(\mathrm{Spec}\, k, \mu_n) \arrow[d, equal] \\
H^1(\mathrm{Spec}\, k, E') \arrow[r] & H^2(\mathrm{Spec}\, k, \mu_n)
\end{tikzcd}$$

$$x \mapsto C_x$$

<u>Lemma</u>. C_x splits $x \cup u$.

<u>Thm</u> (A.–Auel). Let $d = 2, \ldots, 9$, X be cyclic SB var. of degree d over a global field ($d = 8$ or 9, ζ_8, ζ_9).

Then, $p = P_X$ is split by a g1c.

Proof $d=7$.

$$E_\lambda' = y^2 + (1+\lambda-\lambda^2)xy + \lambda^2(1-\lambda)y = x^3 + \lambda^2(1-\lambda)x^2$$

$P=(0,0)$ is a 7-torsion point.

Tom Fisher.

$$u_\lambda = \lambda^3(\lambda-1)^6.$$

Cor. $C_{\chi,\lambda}$ splits $(\chi, \lambda^3(\lambda-1)^6)$.

$$(\chi, p) \quad 10^5$$

David Saltman. $(\chi, u) \longrightarrow p_1, \dots, p_r$

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_{p} Br(k_p) \longrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

$\lambda = p_1 \cdots p_r$ (principal)

$k\left(\sqrt[7]{\lambda^3(\lambda-1)^6}\right)$ splits (χ, u).

Albert (1934). $(\chi, u) = (\chi', \lambda^3(\lambda-1)^6)$.

$\square$.

$C \longrightarrow X$

$O(d)$