

Determinants and deformation
theory of perfect complexes

Joint with Max Lieblich

Motivation

$i: X \hookrightarrow X'$ closed immersion

$$K \subset \mathcal{O}_{X'} \quad K^2 = 0$$

$E \in \mathcal{D}(X)$ perfect complex on X

deformation of E to X' :

$$(E', \sigma) \quad E' \in \mathcal{D}^-(X')$$

$$\sigma: E' \otimes^L \mathcal{O}_X \xrightarrow{\sim} E.$$

Facts: (i) $\exists \omega(E) \in \text{Ext}^2(E, E \otimes K)$

which is 0 $\Leftrightarrow \exists$ def. of E

(ii) If $\omega(E) = 0$ then the set of deformations form a tower under $\text{Ext}^1(E, E \otimes K)$

(iii) If $\text{Ext}^1(E, E) = 0$ then $\text{autos} = \text{Ext}^1(E, E \otimes K)$

• $E \rightarrow \det(E)$ line bundle.

• $\text{tr}: \text{Ext}^1(E, E \otimes K) \rightarrow H^1(X, K)$

[III, V, 3.7.3].

Thm: (i) $\text{tr}(\omega(E)) = \omega(\det(E))$

(ii) $(E', \sigma) \in \text{Ext}^i(E, E \otimes K)$

$$\det(\alpha * (E', \sigma)) = \text{tr}(\alpha) * (\det(E'), \det(\sigma))$$

(iii) —

• Sdruing, Töen, Vezzosi
derived AG (char 0).

(iii) Gabber
deformation theory
of bounded complexes.

Remarks:

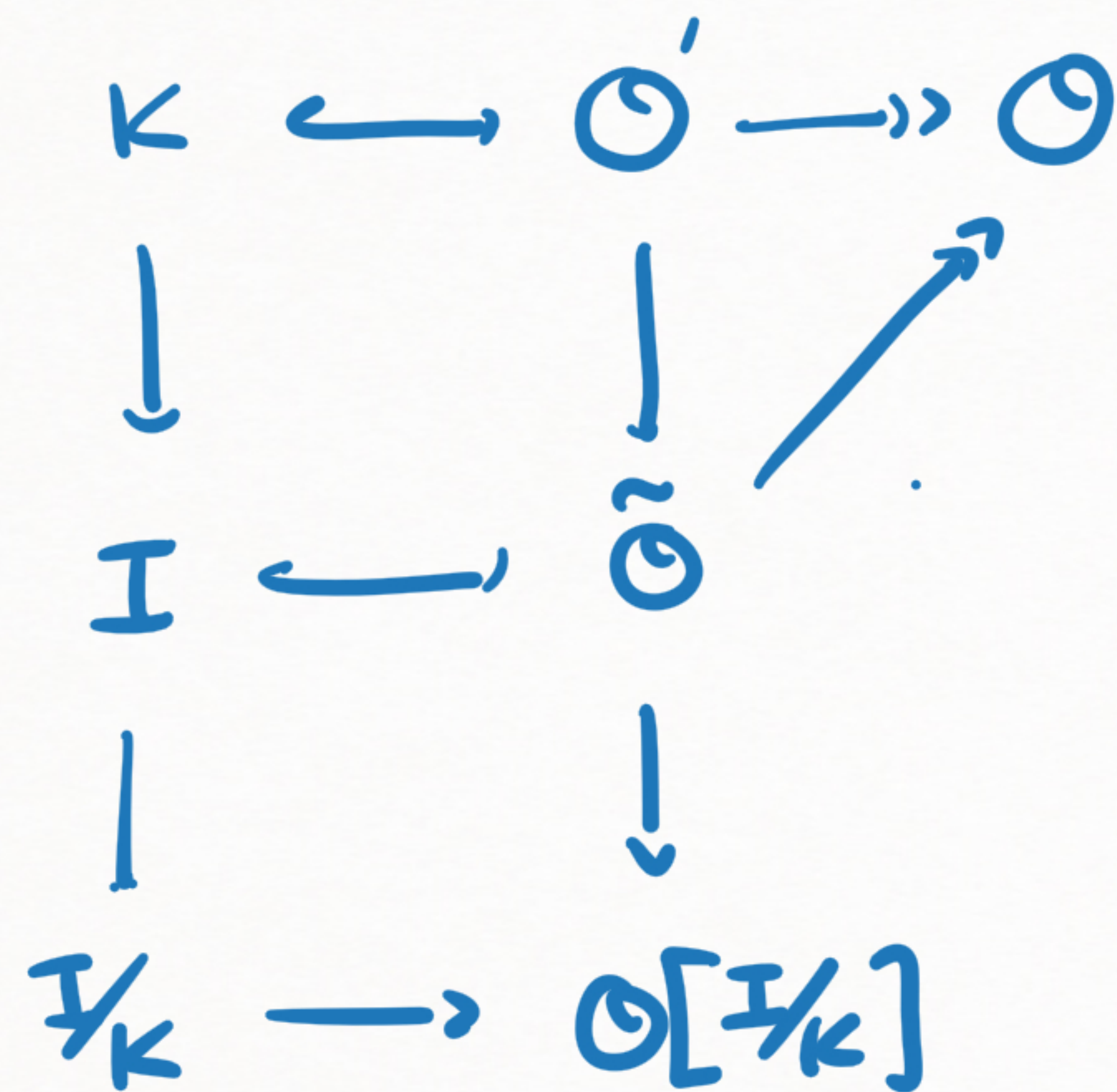
(i) general site $\mathcal{O}' \rightarrow \mathcal{O}$
C

(ii) Spinor curves:
• global resolution
Huybrecht, Thomas
[III, V, 3.7.7]

Some naive ideas

C $K \subset \mathcal{O}' \rightarrow \mathcal{O}$
E

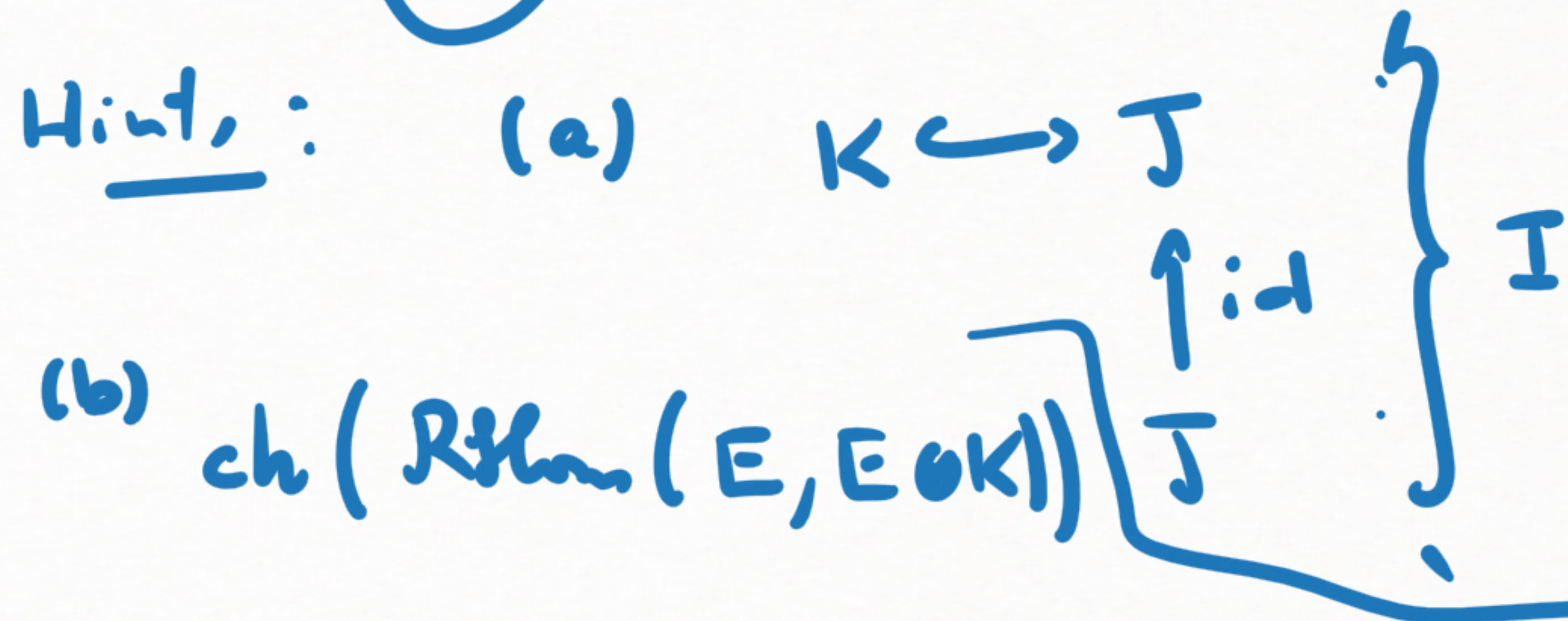
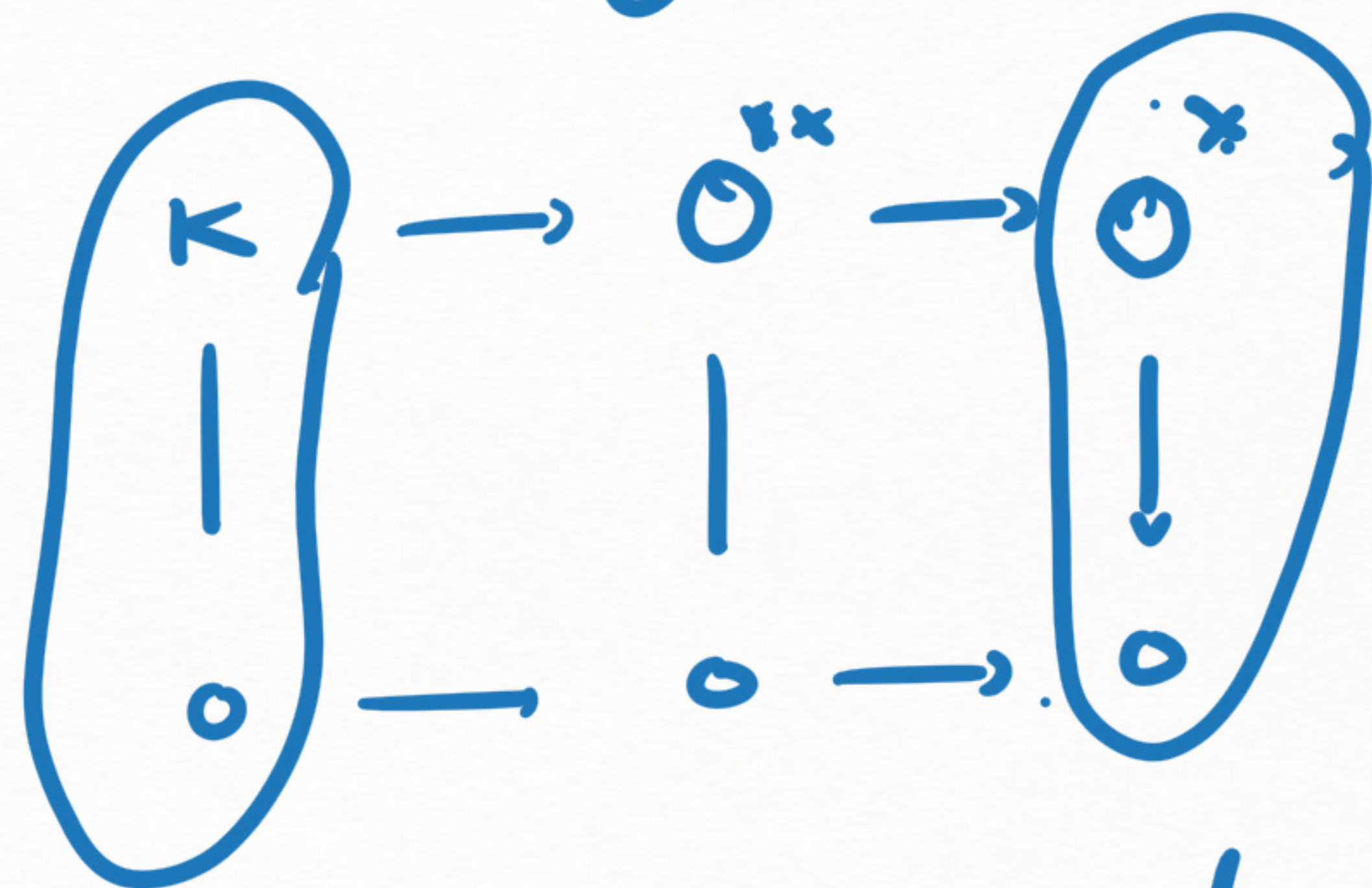
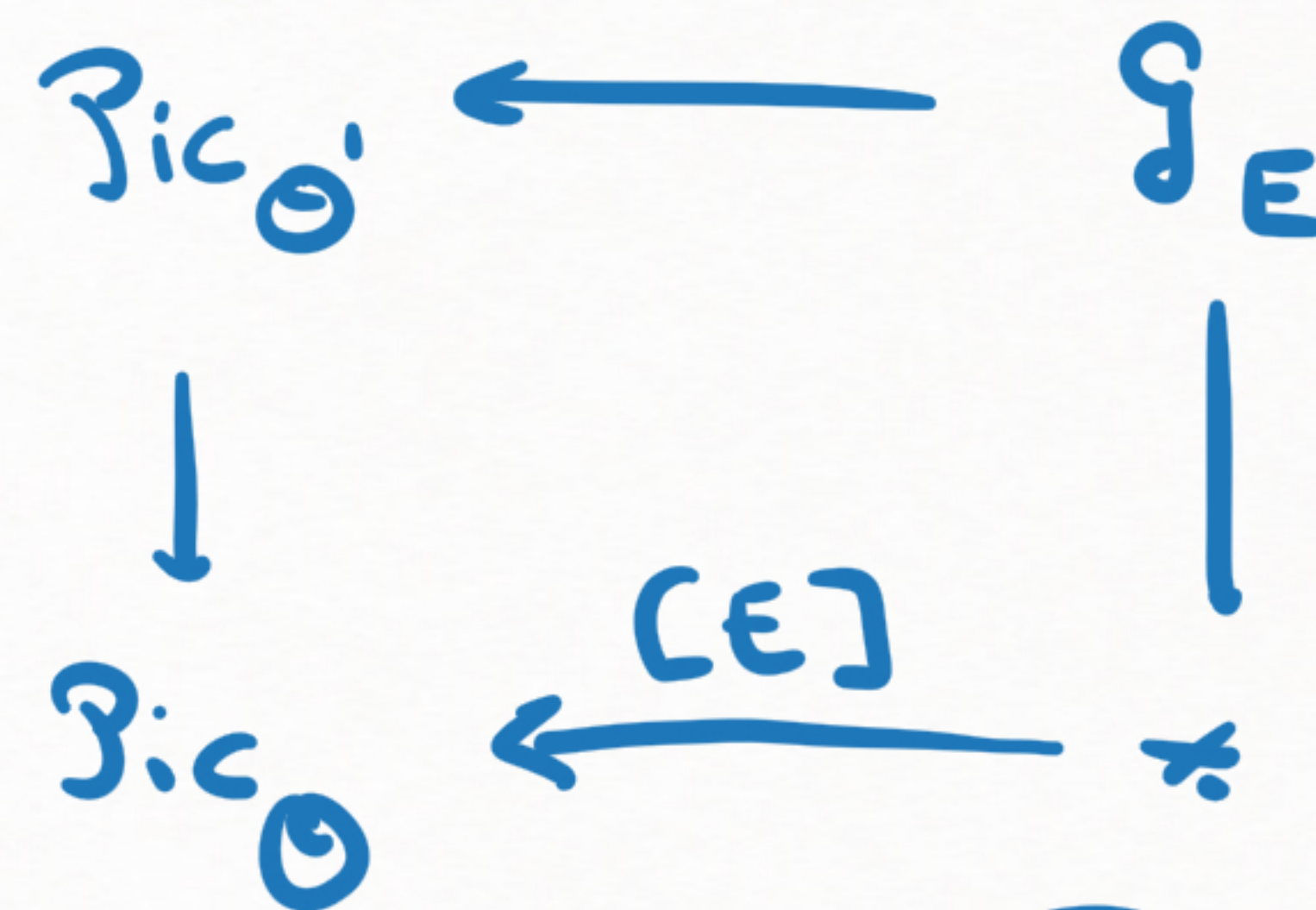
(a) Suppose $\exists K \hookrightarrow I$
s.t. $\text{Ext}^i(E, E \otimes I) = 0$
 $i > 0.$



$$\text{Ext}^2(E, E \otimes I/K) \simeq \text{Ext}^2(E, E \otimes K)$$

Choose lifting \tilde{E} of E to \mathcal{O}
 $\sim \tilde{E} / \mathcal{O}[I/K]$

(b) note 1



$\tilde{L}^{-1} \rightarrow L^0$
 pch(L) objects: $x \in L^0$
 morphism: $x \rightarrow x'$
 $y \in \tilde{L}^{-1}, dy = x' - x.$
 stratifying: $cl(L).$

The ∞ -category of perfect complexes

C site

\mathcal{O}

B' s.c.dga [SP, Tag 06IV]

Mod_B^{dg} [SP, Tag 07JI].

flat model cat structure.
 Lin-Zheng.

$\mathcal{D}(C, B') := N_{dg}(Mod_B)$

[HA, 1.3.1.6].

cat-fib
objects.

$\mathcal{D}_{perf}(C, B') \subset \mathcal{D}(C, B').$

A. simplicial ring.

$\mathcal{D}(C, A.) := \mathcal{D}(C, N(A.))$

Deformations:

$$B \rightarrow C \text{ surj}$$

$$I \quad I^2 = 0.$$

$$E \in \text{Mod}_C^0$$

$$\begin{array}{ccc} & & \mathcal{D}(C, B) \\ & & \downarrow \circlearrowleft_{B'} \quad (*) \\ * & \xrightarrow{[E]} & \mathcal{D}(C, C) \end{array}$$

Thm: Suppose \exists lifting of E to B' . Then fiber product of (*) is given by

$$\text{DK} \left(\tau_{\mathcal{D}}(\text{RHom}(E, E \otimes I)[1]) \right)$$

↙ Dold-Kan.

determinant

$C = \text{pt.}$ $A.$ simplicial ring.

$$\coprod_{n \geq 0} BGL_n(A.) \longrightarrow \mathcal{D}_{\text{perf}}(A.)^{\text{pic}}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$K(\coprod_{n \geq 0} BGL_n(A.)) \xrightarrow{\sim} K(A.)$$

det

det

det

det

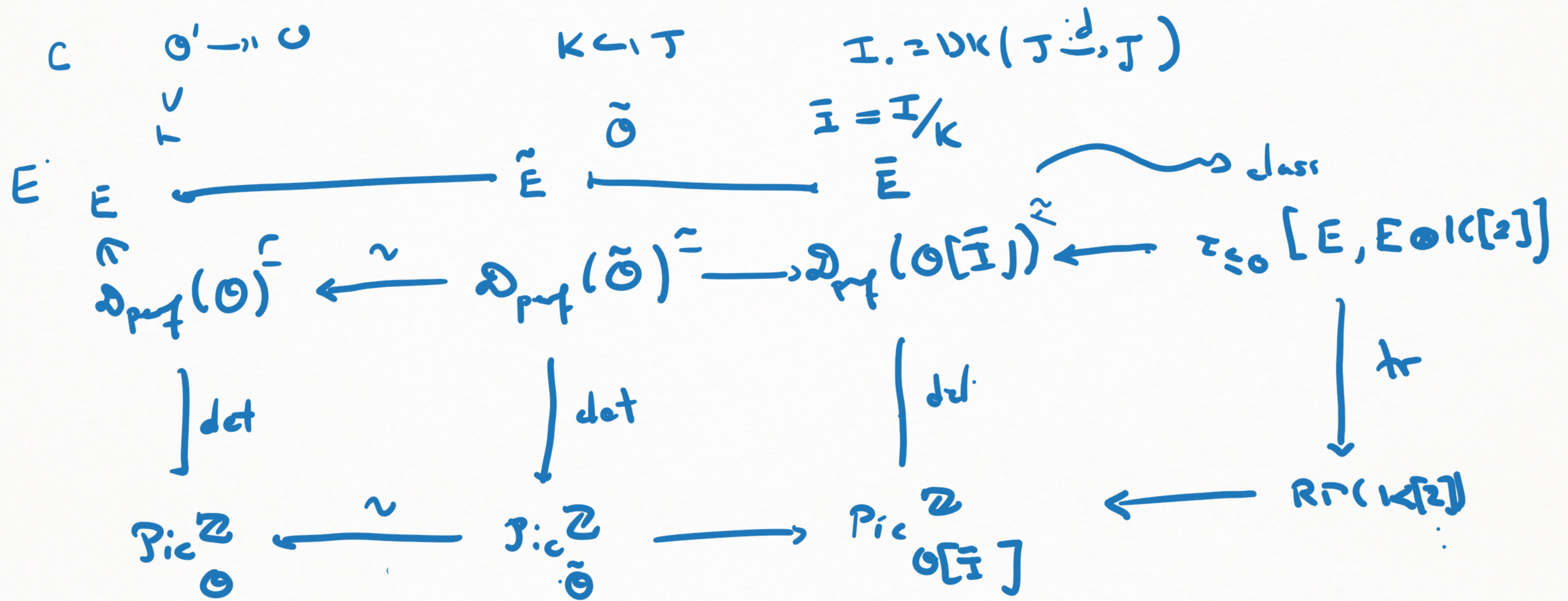
$\text{Pic}^{\mathbb{Z}}(A.)$

If $A. = R$
 pairs (n, L)
 $n \in \mathbb{Z}, L \text{ rkl}$
 module.

$$GL_n(A.) \longrightarrow M_n(A.)$$

$$\downarrow \quad \cong \quad \downarrow$$

$$GL_n(\pi_0(A.)) \longrightarrow M_n(\pi_0(A.))$$



$$\text{tr}(w(E)) = w(\det(E)) \quad \checkmark$$