Determinants and deformation theory of perfect complexes

Joint with Max Lieblich

Motivation

$i : X \hookrightarrow X'$ closed immersion

$K \subset \mathcal{O}_{X'}$ $\quad K^2 = 0$

$E \in D(X)$ perfect complex on X

deformation of E to X':

(E', σ) $\qquad E' \in D^-(X')$

$\sigma : E' \otimes^{\mathbb{L}} \mathcal{O}_X \xrightarrow{\simeq} E$.

Facts: (i) $\exists\ \omega(E) \in \mathrm{Ext}^2(E, E \otimes K)$ which is $0 \iff \exists$ def. of E

(ii) If $\omega(E) = 0$ then the set of deformations form a torsor under $\mathrm{Ext}^1(E, E \otimes K)$.

(iii) If $\mathrm{Ext}^{-1}(E, E) = 0$ then auto's $= \mathrm{Ext}^0(E, E \otimes K)$

- $E \rightsquigarrow \det(E)$ line bundle.
- $\mathrm{tr} : \mathrm{Ext}^i(E, E \otimes K) \to H^i(X, K)$

[Ill, V, 3.7.3].

Thm: (i) $\mathrm{tr}(\omega(E)) = \omega(\det(E))$

(ii) (E', σ) $\alpha \in \mathrm{Ext}^1(E, E \otimes K)$

$$\det(\alpha * (E', \sigma)) = \mathrm{tr}(\alpha) * (\det(E'), \det(\sigma))$$

(iii) —.

Remarks:

(i) general site $\mathcal{C}$ $\quad \mathcal{O}' \to \mathcal{O}$

(ii) Spin curves:

- global resolution

  Huybrechts, Thomas

[Ill, V, 3.7.7]

- Schürg, Toën, Vezzosi: derived AG (char 0).

(iii) Gabber

deformation theory of bounded complexes.

Some naive ideas

$\mathcal{C}$ $\quad K \subset \mathcal{O}' \twoheadrightarrow \mathcal{O}$

E

(a) Suppose $\exists\ K \hookrightarrow I$

s.t. $\mathrm{Ext}^i(E, E \otimes I) = 0$

$i > 0$.

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d] & \mathcal{O}' \arrow[r, two heads] \arrow[d] & \mathcal{O} \\ I \arrow[r, hook] \arrow[d] & \tilde{\mathcal{O}} \arrow[d] \arrow[ur] & \\ I/K \arrow[r] & \mathcal{O}[I/K] & \end{tikzcd}$$

$\mathrm{Ext}^2(E, E \otimes I/K) = \mathrm{Ext}^2(E, E \otimes K)$

$\omega(E)$

choose lifting $\tilde{E}$ of E to $\tilde{\mathcal{O}}$

$\rightsquigarrow \bar{E} \,/\, \mathcal{O}[I/K]$

(b) rk 1

$$\begin{tikzcd} \mathrm{Pic}_{\mathcal{O}'} \arrow[d] & \mathfrak{g}_E \arrow[l] \arrow[d, dash] \\ \mathrm{Pic}_{\mathcal{O}} & * \arrow[l, "{[E]}"'] \end{tikzcd}$$

$$\begin{tikzcd} K \arrow[r] \arrow[d, dash] & \mathcal{O}'^{\times} \arrow[r] \arrow[d, dash] & \mathcal{O}^{\times} \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Hint: (a) $\left.\begin{matrix} K \hookrightarrow J \\ \uparrow \mathrm{id} \\ J \end{matrix}\right\} I$

(b) $\mathrm{ch}(R\mathcal{H}om(E, E \otimes K))$

$L^{-1} \longrightarrow L^0$

$\mathrm{pch}(L^\cdot)$ objects: $x \in L^0$

morphism: $x \longrightarrow x'$

$y \in L^{-1}$, $dy = x' - x$.

stackify: $\mathrm{cl}(L^\cdot)$.

The ∞-category of perfect complexes

C site

$\mathcal{O}$

$B^\cdot$ s.c.dga [SP, Tag 061V]

$\mathrm{Mod}^{dg}_{B^\cdot}$ [SP, Tag 09JI].

flat model cat structure.

Lin-Zhang.

$\mathcal{D}(C, B^\cdot) := N_{dg}(\mathrm{Mod}^{\circ}_{B^\cdot})$

[HA, 1.3.1.6].

cof-fib objects.

$\mathcal{D}_{perf}(C, B^\cdot) \subset \mathcal{D}(C, B^\cdot)$.

$A_\cdot$ simplicial ring.

$\mathcal{D}(C, A_\cdot) := \mathcal{D}(C, N(A_\cdot))$

Deformations:

$B^\cdot \to C^\cdot$ surj

$\cup$

$I^\cdot \qquad I^2 = 0.$

$E \in \mathrm{Mod}^0_{C^\cdot}$

$$
\begin{tikzcd}
 & \mathcal{D}(C, B^\cdot) \arrow[d, "\otimes_{B^\cdot} C^\cdot"] & (*) \\
* \arrow[r, "{[E]}"] & \mathcal{D}(C, C^\cdot) &
\end{tikzcd}
$$

Thm: Suppose $\exists$ lifting of E to $B^\cdot$. Then fiber product of $(*)$ is given by

$$DK\left(\tau_{\leq 0}\left(R\mathrm{Hom}(E, E \otimes I)[1]\right)\right)$$

DK = Dold-Kan.

<u>determinant</u> $C = pt.$ $A_\cdot$ simplicial ring.

$$\begin{tikzcd}
\coprod_{n\geq 0} BGL_n(A_\cdot) \arrow[rr] \arrow[dr] \arrow[ddr, bend right, "\det"'] & & \mathcal{D}_{perf}(A_\cdot)^{\simeq} \arrow[d] \\
& K(\coprod_{n\geq 0} BGL_n(A_\cdot)) \arrow[r, "\text{``}\sim\text{''}"] \arrow[d, "\det"] & K(A_\cdot) \arrow[dl, bend left, "\det"] \\
& \mathrm{Pic}^{\mathbb{Z}}(A_\cdot) &
\end{tikzcd}$$

det

If $A_\cdot = R$ pairs (n, L) $n \in \mathbb{Z}$, L rk 1 module.

$$\begin{tikzcd}
GL_n(A_\cdot) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M_n(A_\cdot) \arrow[d] \\
GL_n(\pi_0(A_\cdot)) \arrow[r] & M_n(\pi_0(A_\cdot))
\end{tikzcd}$$

$C \qquad \mathcal{O}' \twoheadrightarrow \mathcal{O} \qquad\qquad K \hookrightarrow J \qquad\qquad I_0 = \ker(J \xrightarrow{d} J)$

$\tilde{\mathcal{O}} \qquad\qquad \bar{I} = I/K$

E

$$\begin{tikzcd}
E \arrow[d, phantom, "\in" rotate=90] & \tilde{E} \arrow[l, mapsto] \arrow[r, mapsto] & \bar{E} \arrow[rr, squiggly, "\text{class}"] & & \\
\mathcal{D}_{perf}(\mathcal{O})^{\simeq} \arrow[d, "\det"] & \mathcal{D}_{perf}(\tilde{\mathcal{O}})^{\simeq} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\det"] & \mathcal{D}_{perf}(\mathcal{O}[\bar{I}])^{\simeq} \arrow[d, "\det"] & \tau_{\leq 0}[E, E \otimes K[2]] \arrow[l] \arrow[d, "tr"] \\
\mathrm{Pic}^{\mathbb{Z}}_{\mathcal{O}} & \mathrm{Pic}^{\mathbb{Z}}_{\tilde{\mathcal{O}}} \arrow[l, "\sim"'] \arrow[r] & \mathrm{Pic}^{\mathbb{Z}}_{\mathcal{O}[\bar{I}]} & R\Gamma(K[2]) \arrow[l]
\end{tikzcd}$$

$tr(w(E)) = w(\det(E))$ ✓