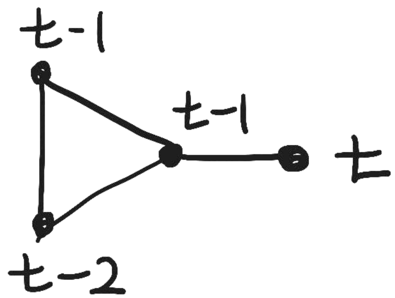
Simplicial generation of Chow rings of matroids

Chris Eur (w/ Spencer Backman & Connor Simpson)

Graph

G :



chromatic polynomial

$\chi_G(t) =$ # proper colorings of G w/ $\leq t$ colors

$$= t(t-1)^2(t-2) = t(\underline{t^3 - 4t + 5t - 2})$$

matroid $M = (E, \mathcal{I})$

E ← ground set; $\mathcal{I}$ ← indep. subsets

E.g. ① $\boxed{E = \{v_0, \dots, v_n\} \text{ spanning } V \simeq \mathbb{C}^{d+1} \ (\text{i.e. } \mathbb{C}^{d+1} \twoheadrightarrow V)}$ $(v_i \neq 0)$

② $G \rightsquigarrow E = \{\text{edges}\}, \quad \mathcal{I} = \text{acyclics}$

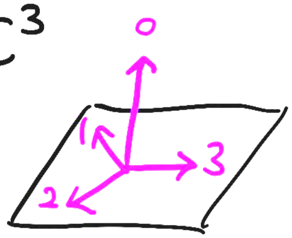
flat: $F \subseteq E$ such that $\text{span}(F \cup \{x\}) \supsetneq \text{span}(F) \ \forall x \notin F$.

hyperplane arr. $\mathcal{A}_M = \{L_i \subset \mathbb{P}V^*\}$, $L_i = \{f \in \mathbb{P}V^* \mid v_i(f) = 0\}$.

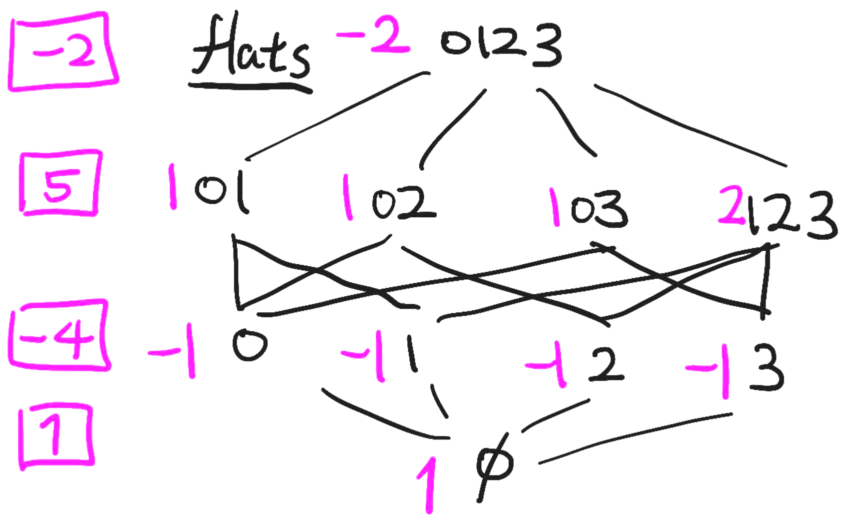
$L_F = \{f \in \mathbb{P}V^* \mid v_i(f) = 0 \ \forall i \in F\}$.

E.g.

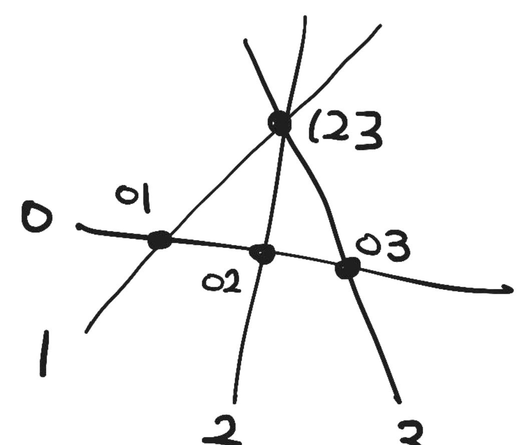
$\mathbb{C}^3$



flats



$\mathcal{A}_M \subset \mathbb{P}^2$



$\chi_M = t^3 - 4t^2 + 5t - 2$

Conj. (Heron-Rota-Welsh 70's)

|coeff's| of $\chi_M(t)$ form log-concave sequence

$$a_i^2 \geq a_{i-1}a_{i+1}$$

Geom. of wonderful cpt. $(\mathbb{C}^{n+1} \twoheadrightarrow V \quad \longleftrightarrow \quad \mathbb{P}V^* \hookrightarrow \mathbb{P}^n)$

$$\begin{tikzcd} X_M \arrow[r, hook] \arrow[d] & X_{A_n} \arrow[d] \\ \mathbb{P}V^* \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

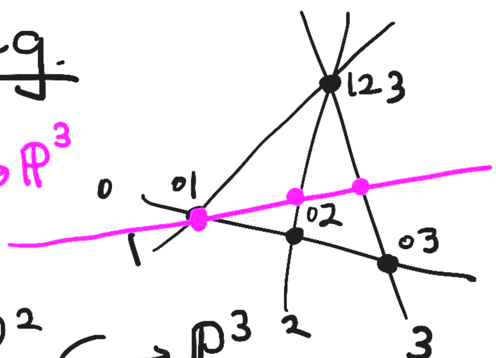
X_{A_n} (permutohedral variety)

blow-up coordinate subspaces (starting w/ pts)

blow-up LF's, starting w/ smallest dim'l ones.

E.g.

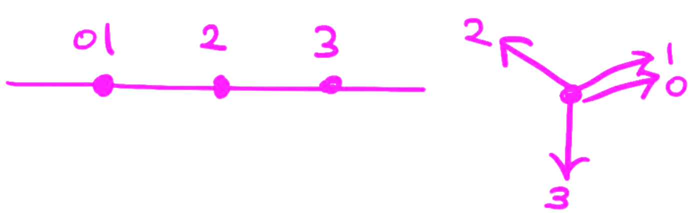
$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$



$\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$

blow-up the fours pts

(then the strict transf. of lines)



Defn/Thm Chow ring of M, $A^\bullet(M) = \bigoplus_{i=0}^{d} A^i(M)$

de Concini -Procesi '95

Feichtner-Yuzvinsky '04

$$= \frac{\mathbb{R}[\,z_F \mid F \subseteq E \text{ nonempty flat}\,]}{\langle z_F z_{F'} \mid F, F' \text{ incomparable} \rangle + \langle \sum_{i \in F} z_F \mid i \in E \rangle}$$

$$\rightarrow A^\bullet(M) = A^\bullet(X_M)$$

$F \subsetneq E$: z_F = exceptional divisor from blowing up L_F.

$-z_E$ = hyperplane class in $\mathbb{P}V^*$ (pullback).

Thm (Adiprasito–Huh–Katz '18) $A^\bullet(M)$ satisfies:

(PD) Poincaré duality

(HL) hard Lefschetz

(HR) Hodge–Riemann $\rightsquigarrow$ $\begin{bmatrix} a_{i-1} & a_i \\ a_i & a_{i+1} \end{bmatrix}$ has $\begin{bmatrix} + & \\ & - \end{bmatrix}$

$a_i^2 > a_{i-1} a_{i+1}$.

Defn Simplicial presentation $A^\bullet_\nabla(M) = \mathbb{R}[h_F\text{'s}]/\langle \sim \rangle$

$$h_F := \sum_{G \supseteq F} -z_G$$

$$H^0(\mathcal{O}(h_F), X_M) = \{\text{hyperplanes in } \mathbb{P}V^* \text{ containing } L_F\}$$

$$h_{01} = -z_E - z_{01} = \widetilde{H} - E_{01}$$

$$\left(\text{i.e. } h_F = \text{hyperplane pullback from } \begin{tikzcd} \mathbb{P}V^* \arrow[rr, dashed] \arrow[dr] & & \mathbb{P}(V^*/L_F) \\ & X_M \arrow[ur] & \end{tikzcd} \right)$$

$\boxed{\text{KEY 1.}}$ variable $h_F \longleftrightarrow$ principal truncation of M.

$\rightsquigarrow$ a monomial basis in $h_F \longleftrightarrow$ relative Schubert matroids.

$\rightsquigarrow$ **Cor** Recover Poincaré duality of $A^\bullet(M)$.

X d-dim'l smth proj. $\mathbb{C}$-var,

$D_1, \dots, D_\ell \in (\mathrm{Pic}\, X)_{\mathbb{R}}$ b.p.f. (nef), $\int_X : A^d(X) \longrightarrow \mathbb{R}$

$$VP_X(\underline{t}) = \int_X (t_1 D_1 + \cdots + t_\ell D_\ell)^d \in \mathbb{R}[t_1, \dots, t_\ell]$$

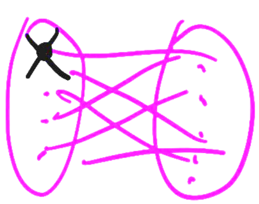
<u>Thm</u> VP_X has non-neg. coeff & log-concave on $\mathbb{R}^\ell_{>0}$.

$\boxed{\text{KEY 2.}}$ h_F is b.p.f.

<u>Thm</u> $\int_M : A^d(M) \longrightarrow \mathbb{R}$. For multiset of size d $\{F_1, \dots, F_d\}$

$$\int_M h_{F_1} \cdots h_{F_d} = \begin{cases} 1 & \text{if } \boxed{\begin{array}{l} \mathrm{rk}_M\left(\bigcup_{j \in J} F_j\right) \geq |J| + 1 \\ \text{for all } \emptyset \neq J \subseteq \{1, \dots, d\}. \end{array}} \\ 0 & \text{else} \end{cases}$$

$(\mathrm{rk}(S) = \dim \mathrm{span}(S))$



matching problem $\leftarrow$ Hall — Hall-Rado

(Postnikov) dragon Hall — dragon Hall-Rado

Thm $VP_M^{\nabla}(t_F's) = \int_M \left(\sum_{F\subseteq E} t_F h_F\right)^d$ is log-conc.

on positive-orthant.

$\Uparrow$

VP_M^{∇} is Lorentzian (Brändén–Huh '18)

(Anari–Liu–Oveis Gharan–Vinzant '18)

$\downarrow$

support

partial deriv.

$\rightsquigarrow$ (HR) in degree 1.

More Koszul ?

VP_M in z_p's vs. VP_M^{∇} ?

Other (aug. / conormal) ?