Symbols, birational geometry, and computations

Stanford Algebraic Geometry Seminar

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Guiding questions and background

Fundamental question

G finite group

Suppose that G acts birationally and faithfully on a complex variety – after resolving indeterminacy and singularities we may assume G acts regularly on a smooth projective variety X.

Classify such actions $X \circlearrowleft G$ up to birational conjugation, especially in cases where $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$, i.e., realizations of G in the *Cremona group* $\mathrm{Bir}(\mathbb{P}^n)$

A sampling of existing results

Restricting to work in my lifetime:

- Iskovskikh laid the groundwork with the G-birational classification of surfaces and their linkings;
- Beauville, Bayle, de Fernex, and Blanc classify actions for finite abelian G on surfaces;
- Dolgachev and Iskovskikh largely completed the surface case;
- Bogomolov and Prokhorov consider the stable conjugacy problem for the surface case using cohomological tools;
- Prokhorov, Shramov, and collaborators have numerous theorems for threefolds – both for specific groups (like simple groups) and general structural results.

Some observations

Much of this work fits into the minimal model program, using distinguished models to reduce the classification problem to an analysis of automorphisms of a restricted class of objects. For example, to classify G-actions on rational surfaces it suffices to look at 'minimal cases': $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, and other del Pezzo surfaces, along with conic bundles $X \to \mathbb{P}^1$.

With a few exceptions – the application of the cohomology on the Néron-Severi group by Bogomolov-Prokhorov and the 'normalized fixed curve with action (NFCA)' invariant of Blanc – invariants play a limited role.

Introducing the invariants

Kontsevich-Peshtun-Tschinkel invariants for $G = C_p$

Let X be a smooth projective variety with a generically free action $X \circlearrowleft C_p$ of a cyclic group of prime order. Consider the fixed points

$$X^{C_p} = \coprod_{\alpha} F_\alpha,$$

a disjoint union of smooth closed subvarieties. Write $[a_{1,\alpha}, \ldots, a_{n,\alpha}]$ for the weights for the action of C_p on $T_{x_\alpha}X$ for some $x_\alpha \in F_\alpha$. The number of zero weights is $\dim(F_\alpha)$. We seek to define

$$\beta(X \circlearrowleft C_p) = \sum_{\alpha} [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

but must introduce relations to get a well-defined birational invariant.

Examples

Let C_p act diagonally on $\mathbb{P}^2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^s \end{pmatrix} \quad \zeta = e^{2\pi i/p}, 0 < s < p$$

with symbols

$$\beta(\mathbb{P}^2 \circlearrowleft C_p) = [1, s] + [-1, s-1] + [-s, 1-s],$$

understood as integers mod p.
Let C_2 act on the degree-two del Pezzo surface $X = \{w^2 = f_4(x, y, z)\}$ via $\pm w$. We get symbols

$$\beta(X \circlearrowleft C_2) = [1, 0].$$

Algebra of symbols

Let G be an abelian group, $A = \mathrm{Hom}(G, \mathbb{G}_m)$, $n \in \mathbb{N}$. Consider the free abelian group on symbols $[a_1, \dots, a_n]$, where $\{a_1, \dots, a_n\}$ generate A. Let $\mathcal{B}_n(G)$ denote the quotient under the relations:

- for each $\sigma \in \mathfrak{S}_n$ we have

$$[a_{\sigma(1)}, \dots, a_{\sigma(n)}] = [a_1, \dots, a_n];$$

- for all $2 \leqslant k \leqslant n$, $a_1, \dots, a_k \in A$, and $b_1, \dots, b_{n-k}$ with $\{a_1, \dots, a_k, b_1, \dots, b_{n-k}\}$ generating A, we write

$$[a_1, \dots, a_k, b_1, \dots, b_{n-k}] = \sum_{1 \leqslant i \leqslant k, a_i \neq a_{i'} \text{ for } i' < i} [a_1 - a_i, \dots, \underbrace{a_i}_{i-\text{th}}, \dots, a_k - a_i, b_1, \dots, b_{n-k}].$$

The latter reflects blowing up a G-stable stratum; when $k = n$ this is a fixed point and we keep track what's fixed in the exceptional divisor.

Evaluating invariants

Let $\dim(X) = n$ and G be abelian. Consider $X \circlearrowleft G$ and identify fixed points

$$X^G = \coprod_\alpha F_\alpha$$

as before. Write $[a_{1,\alpha}, \ldots, a_{n,\alpha}]$ for the weights of G on $T_{x_\alpha} X$ for some $x_\alpha \in F_\alpha$. We express

$$\beta(X \circlearrowleft G) = \sum_\alpha [a_{1,\alpha}, \ldots, a_{n,\alpha}] \in \mathcal{B}_n(G).$$

We ignore points with nontrivial stabilizer $H \subsetneq G$.

Reichstein and Youssin developed a similar invariant taking into account the *determinant* character on the tangent bundle.

First examples

Curves

We have

$$\mathcal{B}_1(G) = \begin{cases} \mathbb{Z}^{\phi(N)} & \text{if } G = C_N \\ 0 & \text{otherwise} \end{cases}$$

with basis $[a]$ where a is relatively prime to N. (We identify $\mathrm{Hom}(C_N, \mathbb{G}_m) = \mathbb{Z}/N\mathbb{Z}$.)
On $\mathbb{P}^1$, the only generically-free action is

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta^a \end{pmatrix} \quad \zeta = e^{2\pi i/N}, (a, N) = 1$$

with symbol $[a] + [-a]$. Imposing the additional relation $[-a] \equiv -[a]$ coincides with setting $\mathbb{P}^1 \circlearrowleft C_N$ equal to zero.
Over higher genus curves, the Riemann existence theorem allows us to produce arbitrary symbols.

A construction theorem

Theorem

Let p be prime. Then $\mathcal{B}_n(C_p)$ is generated as an abelian group by $\beta(X \circlearrowleft C_p)$ where X is smooth and projective.

Induction on n, with base case of curve described above.
For $[a_1, \dots, a_{n-1}, 0]$ we construct $(n-1)$-dimensional varieties D with the prescribed invariants and $D \times \mathbb{P}^1$ with trivial action on $\mathbb{P}^1$.
Since $[a, a, a_3, \dots, a_n] = [0, a, a_3, \dots, a_n]$ we may focus on symbols $[a_1, a_2, \dots, a_n], 0 < a_1 < a_2 < \dots < a_n < p$. Any expression

$$\sum m_{[a_1,a_2,\dots,a_n]}[a_1, a_2, \dots, a_n], m_{[a_1,a_2,\dots,a_n]} \geqslant 0,$$

is realized as $\beta(X \circlearrowleft C_p)$, where X is smooth, projective, and irreducible. There is a direct construction as a complete intersection of high degree.

Linear actions

Suppose that C_N acts linearly on $\mathbb{P}^n$

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \zeta^{a_1} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \zeta^{a_n} \end{pmatrix}, \quad \zeta = e^{2\pi i/N}.$$

If the a_i are distinct and non-trivial then the symbol is

$$[a_1, \dots, a_n] + [-a_1, a_2 - a_1, \dots, a_n - a_1] + \dots + [-a_n, a_1 - a_n, \dots, a_{n-1} - a_n].$$

This is *not necessarily* zero in $\mathcal{B}_n(C_N)$ but if we assume that we can take signs out across symbols this is equal to

$$[a_1, \dots, a_n] - [a_1, a_2 - a_1, \dots, a_n - a_1] - \dots - [a_n, a_1 - a_n, \dots, a_{n-1} - a_n],$$

one of our relations.

Surfaces

Invariants in low dimensions

$\dim \mathcal{B}_2(C_N) \otimes \mathbb{Q}$ is given by

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	0	1	1	2	2	3	3	5	4	6	7	8	7	13	10

and for primes $p \geqslant 3$ we have $\mathcal{B}_2(C_3) \simeq \mathbb{Z}$ and

$$\dim \mathcal{B}_2(C_p) \otimes \mathbb{Q} = \frac{p^2 + 23}{24}.$$

Note that the genus of the modular curve

$$g(X_1(p)) = \frac{p^2 + 23}{24} - \frac{p-1}{2}.$$

$\mathcal{B}_2(C_2 \times C_2) \simeq C_2 \times C_2$
$\dim \mathcal{B}_2(C_N) \otimes \mathbb{F}_2$ is given by

N	2	3	4	5	6	7	8
	0	1	1	2	3	4	4

Involutions on surfaces

Since $\mathcal{B}_2(C_2) = 0$ this invariant says **nothing** about involutions of surfaces! Bertini, Geiser, and De Jonquières involutions are perhaps the most intricate parts of the classification.

There is a refinement: For the symbols of type $[a, 0]$ corresponding to curves F_α fixed by C_n, we keep track of the stable birational equivalence class $[F_\alpha]$ and the element of $\mathcal{B}_1(C_n)$ associated with $[a]$.

For general n, we get contributions involving $\mathcal{B}_m(G)$ for $m \leqslant n$, where the symbols incorporate stable birational equivalence classes of varieties of dimension $\leqslant n - m$.

Order three actions on surfaces

We have $\mathcal{B}_2(C_3) \simeq \mathbb{Z}$ with $[1,2] \equiv 0$ and

$$[1,1] \equiv [0,1] \equiv -[0,2] \equiv -[2,2].$$

Actions $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft C_3$, have trivial invariant, e.g., for

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \quad \zeta = e^{2\pi i/3}$$

we have $\beta(\mathbb{P}^2 \circlearrowleft C_3) = [1,2] + [2,1] + [1,2] \equiv 0$. Actions with fixed curves F_α

$$\{w^3 = f_3(x,y,z)\} \subset \mathbb{P}^3, \quad \{w^2 = z^3 + g_6(x,y)\} \subset \mathbb{P}(3,1,1,2)$$

have invariant $[1,0] \in \mathcal{B}_2(C_3)$. Taking into account the class $[F_\alpha]$ gives a complete invariant.

High-order automorphisms

$$X = \{w^2 = z^3 + x(x^5 + y^5)\} \subset \mathbb{P}(3,1,1,2)$$

admits an automorphism of order 30

$$(w : x : y : z) \mapsto (-w : x : \zeta_5 y : \zeta_3 z)$$

with fixed point $(0 : 0 : 1 : 0)$ and with weights $[3,2]$, thus $\beta(X \circlearrowleft C_{30}) = [3,2] \neq 0$, and is not conjugate to a linear action.

$$X = \{w^2 = z^3 + xy(x^4 + y^4)\} \subset \mathbb{P}(3,1,1,2)$$

admits an automorphism of order 24

$$(w : x : y : z) \mapsto (\zeta_8 w : x : iy : -i\zeta_3 z).$$

The fixed point is $(0 : 1 : 0 : 0)$ with symbol $[21,22]$ and $\beta(X \circlearrowleft C_{24}) \neq 0$.

Generalization: Kresch-Tschinkel invariants

Preparation: Assume $X \circlearrowleft G$ is faithful, G is not necessarily abelian, but all stabilizers are abelian. This can be achieved by blowing up along the locus with nontrivial stabilizers, with a view toward making it a normal crossings divisor, strict in the sense that conjugate components are disjoint. Technically, we apply the divisorialification algorithm of Bergh-Rydh to the quotient stack $[X/G]$, building on earlier resolution schemes of Reichstein-Youssin.

Construction of the symbols: We consider each stratum with nontrivial stabilizer $H \subset G$, the action of the normalizer $N_G(H)/H$ on (the orbit of) the stratum, and the induced action of H on its normal bundle. And we record the birational types of the strata along with their induced group actions.

The required blowup relations are a bit complicated but similar in spirit to what was written above.

For cyclic groups acting on rational surfaces, we recover the NFCA invariant of Blanc, which governs his classification.

An example

Iskovskikh exhibited two nonconjugate copies of $G = C_2 \times \mathfrak{S}_3$ in $\mathrm{Bir}(\mathbb{P}^2)$:

- the action on $x_1 + x_2 + x_3 = 0$ by permutation and reversing signs, with model $\mathbb{P}^2$;
- the action on $y_1 y_2 y_3 = 1$ by permutation and taking inverses, with model a sextic del Pezzo surface.

Neither model satisfies the stabilizer condition! Blow up points:

- $(x_1, x_2, x_3) = (0, 0, 0)$ with G as stabilizer;
- $(y_1, y_2, y_3) = (1, 1, 1)$, with G as stabilizer, and $(\omega, \omega, \omega), (\omega^2, \omega^2, \omega^2)$, $\omega = e^{2\pi i/3}$, with $\mathfrak{S}_3$ as stabilizer.

Consider $\mathbb{P}^1$'s stabilized by $H = C_2$ with $\mathfrak{S}_3$ action: The first surface has the exceptional divisor and the line at infinity; the second has just the exceptional divisor. Further blow-ups will not introduce new curves of this type.

Threefolds

Invariants for threefolds

$\dim \mathcal{B}_3(C_N) \otimes \mathbb{Q}$ is given by

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	0	0	0	0	0	0	0	1	0	1	2	2	1	5	3

$\dim \mathcal{B}_3(C_N) \otimes \mathbb{F}_p$ is given by

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	15
p=2	0	0	0	0	0	**1**	**1**	1	**2**	1	**4**	2	**6**	**6**	**9**
p=3	0	0	0	0	0	0	0	1	0	1	2	2	1	5	3
p=5	0	0	0	0	0	0	0	1	0	**2**	2	2	1	5	3

where the bolded numbers indicate torsion

Complete intersections of two quadrics

Consider

$$X = \{\sum_{i=0}^{5} x_i^2 = \sum_{i=0}^{5} \lambda_i x_i^2\} \subset \mathbb{P}^5$$

with associated sextic form

$$f_6(t_0, t_1) = \prod_{i=0}^{5} (t_0 + \lambda_i t_1), \quad D = \{f_6 = 0\} \subset \mathbb{P}^1.$$

We have

$$\mathrm{Aut}(X) = C_2^5 \rtimes \mathrm{Aut}(D \subset \mathbb{P}^1)$$

where the first factor acts diagonally by ± 1. Avilov has analyzed these from the standpoint of the G-minimal model program. These should be G-birational to $\mathbb{P}^3$ if and only if they admit a G-invariant line.

The action of C_2 by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is *not* rational but has vanishing invariant, reflecting $\mathcal{B}_2(C_2) = 0$.
The action of $C_2 \times C_2$ on

$$X = \{\sum_{i=0}^{5} x_i^2 = x_0 x_1 + \sum_{i=2}^{5} \lambda_i x_i^2\} \subset \mathbb{P}^5$$

by the matrix above and the interchange of x_0 and x_1 does have

$$\beta(X \circlearrowleft C_2 \times C_2) \neq 0.$$

$\mathrm{PGL}_2(\mathbb{F}_{11})$ threefolds

The first prime p for which $\mathcal{B}_3(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q} \neq 0$ is $p = 11$. Can we realize this via Fano varieties? The antisymmetric matrix

$$\begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} & p_{04} & p_{05} \\ -p_{01} & 0 & p_{04} & 0 & 0 & -p_{02} \\ -p_{02} & -p_{04} & 0 & 0 & p_{03} & 0 \\ -p_{03} & 0 & 0 & 0 & -p_{05} & p_{01} \\ -p_{04} & 0 & -p_{03} & p_{05} & 0 & 0 \\ -p_{05} & p_{02} & 0 & -p_{01} & 0 & 0 \end{pmatrix}$$

has Pfaffian cubic form

$$p_{01}^2 p_{03} + p_{03}^2 p_{02} + p_{02}^2 p_{05} + p_{05}^2 p_{04} + p_{04}^2 p_{01}$$

and the resulting cubic threefold $Y \subset \mathbb{P}^4$ carries an action of $G = \mathrm{PGL}_2(\mathbb{F}_{11})$, compatible with the Pfaffian representation.

There is a dual associated Fano threefold of genus eight $X \subset \mathbb{P}^9$ arising as a codimension-five linear section of the Grassmannian $\mathrm{Gr}(2,6)$ with equations

$$
\begin{aligned}
-u_{35}u_{23} - u_{15}u_{13} - u_{24}u_{12} = -u_{35}u_{24} - u_{15}u_{14} - u_{12}^2 &= 0\\
-u_{35}u_{25} - u_{15}^2 + u_{34}u_{12} = -u_{35}u_{34} + u_{24}u_{14} - u_{12}u_{13} &= 0\\
-u_{35}^2 + u_{24}u_{15} + u_{34}u_{13} = -u_{35}u_{45} + u_{12}u_{15} + u_{34}u_{14} &= 0\\
u_{15}u_{34} + u_{24}^2 - u_{12}u_{23} = u_{15}u_{35} + u_{24}u_{25} + u_{34}u_{23} &= 0\\
u_{15}u_{45} + u_{12}u_{25} + u_{34}u_{24} = -u_{24}u_{45} + u_{12}u_{35} + u_{34}^2 &= 0\\
u_{12}u_{34} - u_{13}u_{24} + u_{14}u_{23} = u_{12}u_{35} - u_{13}u_{25} + u_{15}u_{23} &= 0\\
u_{12}u_{45} - u_{14}u_{25} + u_{15}u_{24} = u_{13}u_{45} - u_{14}u_{35} + u_{15}u_{34} &= 0\\
u_{23}u_{45} - u_{24}u_{35} + u_{25}u_{34} &= 0
\end{aligned}
$$

that also admits a G-action.

The birational geometry of X and Y are tightly intertwined yet distinct:

- X and Y are birationally equivalent;
- X and Y are G-stably birationally equivalent, e.g., via the projective duality construction of A. Kuznetsov;
- X and Y are **not** G birationally equivalent, by work of Cheltsov-Shramov;
- for abelian $H \subset \mathrm{PGL}_2(\mathbb{F}_{11})$ we have

$$\beta(X \circlearrowleft H) = \beta(Y \circlearrowleft H)$$

  and indeed, X and Y are H-birational.

In progress: Distinguish non-abelian actions on X and Y with the Kresch-Tschinkel invariant.

Fourfolds

Cubic fourfolds

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. No examples are known to be irrational! But there are group actions $X \circlearrowleft G$ where rationality fails:

- $\{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_4^2x_5 + x_5^2x_0 + x_0^3 = 0\}$ admits an action of C_{36} with weights $(0, 4, -8, 16, 9, 18)$ with $\beta(X \circlearrowleft C_{36}) \neq 0$.
- $\{x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_0 + x_0^3 + x_1^3 = 0\}$ admits an action of C_{48} with weights $(0, 16, 3, -6, 12, -24)$ with nonvanishing invariant associated with the divisor $x_1 = 0$ stabilized by $C_3 \subset C_{48}$.
- There are Pfaffian (rational!) cubic fourfolds with C_6-action with nonvanishing invariant.