Zeta functions and decomposition spaces

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preliminary joint work with

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Classical zeta functions

	$K/\mathbb{Q}$ # field	$X/\mathbb{F}_q$ variety
Riemann	Dedekind	Hasse-Weil
name:	name:	name:

$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$	$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$
product formula:	product formula:	product formula:
$\zeta_{\mathbb{Q}}(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$	$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$	$Z(X,t) = \prod_{x \in \lvert X \rvert} \left(1 - t^{\deg(x)}\right)^{-1}$
		cycle formula:
		$Z(X,t) = \sum_{\alpha} 1 \, t^{\deg(\alpha)}$
underlying poset:	underlying poset:	underlying poset:
$(\mathbb{N}, \mid)$	$(I_K, \mid)$	$(Z_0^{\text{eff}}(X), \leq)$
arithmetic functions:	arithmetic functions:	arithmetic functions:
$f: \mathbb{N} \to \mathbb{C}$	$f: I_K \to \mathbb{C}$	$f: Z_0^{\text{eff}}(X) \to \mathbb{C}$

$f: \mathbb{N} \to \mathbb{C}$	$f: I_K \to \mathbb{C}$	$f: \mathcal{O} \to \mathbb{C}$
Dirichlet series: $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$	Dirichlet series: $\sum_{a} \frac{f(a)}{N(a)^s}$	Dirichlet series: $\sum_{\alpha} f(\alpha) t^{\deg(\alpha)}$
convolution: $\left(\sum \frac{f(n)}{n^s}\right)\left(\sum \frac{g(n)}{n^s}\right) = \sum \frac{h(n)}{n^s}$	convolution: $\left(\sum \frac{f(a)}{N(a)^s}\right)\left(\sum \frac{g(a)}{N(a)^s}\right) = \sum \frac{h(a)}{N(a)^s}$	convolution: $\left(\sum f(\alpha) t^{\deg(\alpha)}\right)\left(\sum g(\alpha) t^{\deg(\alpha)}\right) = \sum h(\alpha) t^{\deg(\alpha)}$
$h(n) = \sum_{i+j=n} f(i)g(j)$	$h(a) = \sum_{bc=a} f(b)g(c)$	$h(\alpha) = \sum_{\beta+\gamma=\alpha} f(\beta)g(\gamma).$
zeta function: $\zeta: n \mapsto 1$	zeta function: $\zeta: a \mapsto 1$	zeta function: $\zeta: \alpha \mapsto 1$
Möbius function: $\mu: n \mapsto \begin{cases} 0, & p^2 \mid n \\ (-1)^r, & \end{cases}$	Möbius function: $\mu: a \mapsto \begin{cases} \_\_\_ \\ (-1)^r \end{cases}$ $a = p_1 \cdots p_r$	Möbius function: $\mu: \alpha \mapsto$

$n = p_1 \cdots p_r$

Möbius inversion:

$$\mu * \zeta = 1$$

$$\updownarrow$$

$$\zeta_{\mathbb{Q}}(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

Möbius inversion:

$$\mu * \zeta = 1$$

$$\updownarrow$$

$$\zeta_K(s)^{-1} = \sum_{a} \frac{\mu(a)}{N(a)^s}$$

Möbius inversion:

$$\mu * \zeta = 1$$

$$\updownarrow$$

$$Z(X,t)^{-1} = \sum_{\alpha} \mu(\alpha)\, t^{\deg(\alpha)}$$

Zeta functions for posets

An **interval** in a (locally finite) poset $(P, \leq)$ is a sub-poset

$$[x,y] = \{ z \in P \mid x \leq z \leq y \}.$$

DEF The **incidence coalgebra** of a (locally finite) poset $(P, \leq)$ is the free k-vector space $C(P)$ on intervals, with:

$$\Delta : C(P) \longrightarrow C(P) \otimes C(P)$$
$$[x,y] \longmapsto \sum_{z \in [x,y]} [x,z] \otimes [z,y]$$

DEF The **incidence algebra** of $(P, \leq)$ is the dual vector space

$$I(P) = \mathrm{Hom}_k(C(P), k)$$

with multiplication given by convolution:

$$I(P) \otimes I(P) \longrightarrow I(P)$$
$$\varphi \otimes \psi \longmapsto (\varphi * \psi)([x,y]) := \sum_{z \in [x,y]} \varphi([x,z])\, \psi([z,y])$$

$z \in [x,y]$

In $I(P)$, there are distinguished elements:

$$\zeta : C(P) \longrightarrow k$$
$$[x,y] \longmapsto 1$$

$$\mu : C(P) \longmapsto k$$

(recursive formula)

which satisfy $\mu * \zeta = 1 = \zeta * \mu$.

Ex The Riemann, Dedekind and Hasse-Weil zeta functions, and their Möbius inversion principles, are all special cases.

Q. Why do we need to pass to the reduced

Subtlety: need to

incidence algebra, e.g. $I(\mathbb{N}) \rightsquigarrow \widetilde{I}(\mathbb{N})$

$[m,n] \rightsquigarrow [1, \frac{n}{m}]$

Fact: One can deduce the product formulas for $\zeta_{\mathbb{Q}}(s)$, $\zeta_K(s)$ and $Z(X,t)$ from a decomposition of the corresponding posets $\mathbb{N}$, I_K, $Z_0^{\mathrm{eff}}(X)$.

$\boxed{\text{Decomposition spaces}}$

(due to Gálvez-Carrillo — Kock — Tonks)

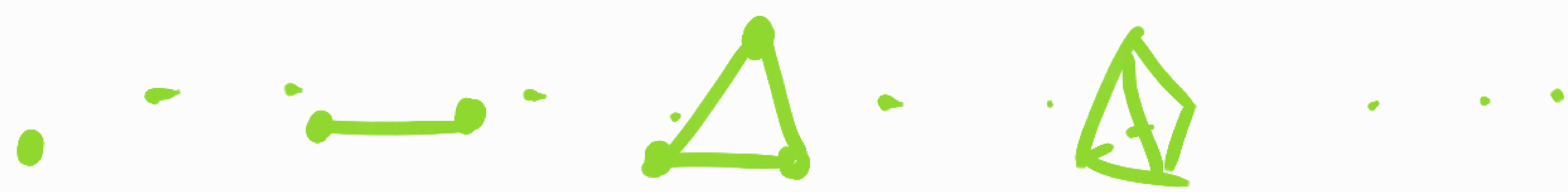
Idea: incidence algebras don't just come from posets, but <u>higher homotopy-theoretic structure</u>.

Recall: a simplicial set is a functor

$$X : \Delta^{op} \longrightarrow \mathrm{Set}$$

cat. of combinatorial simplices

$$X_0 \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow}\; X_1 \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; X_2 \;\cdots\quad - \; - \; -$$

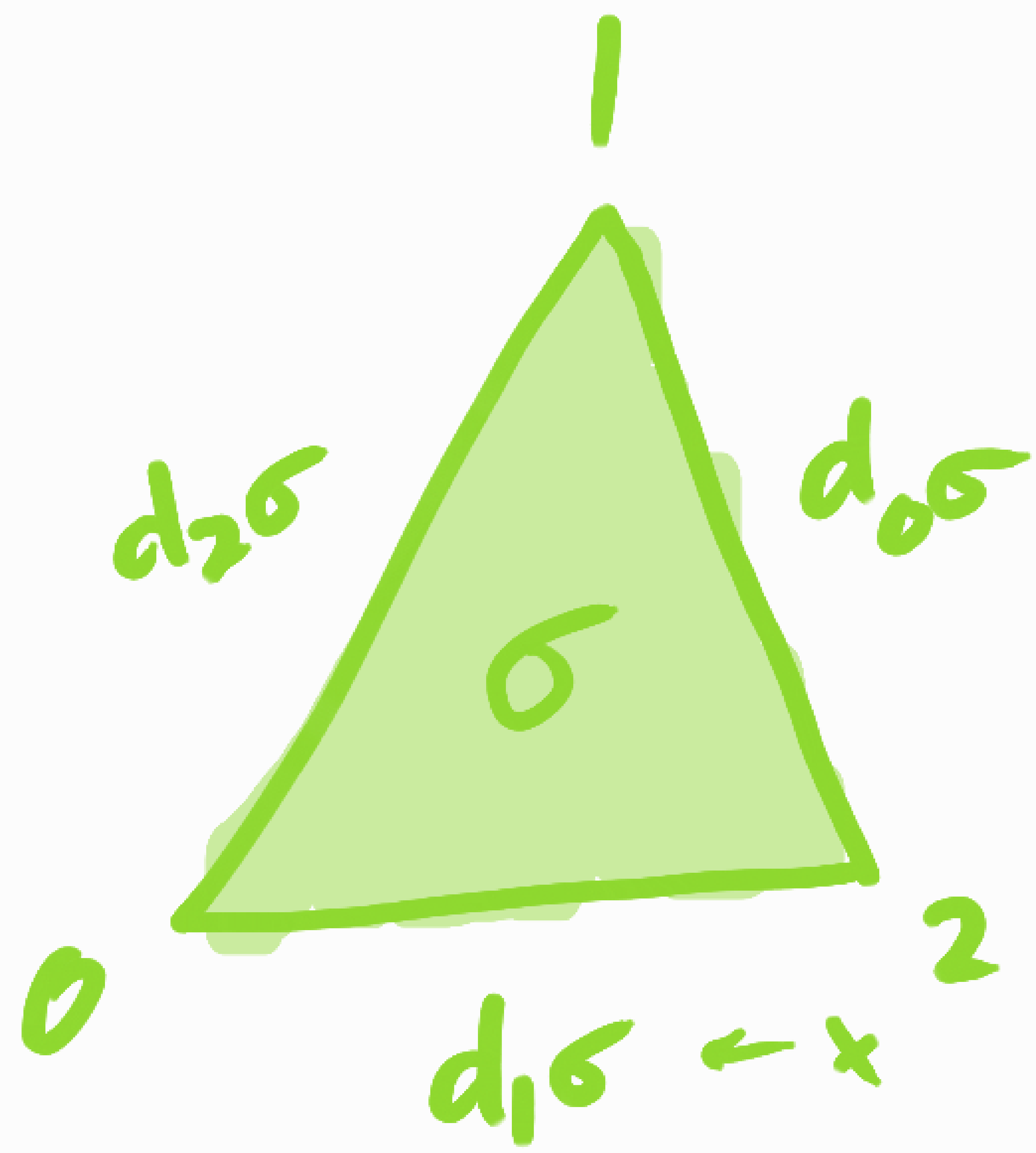


DEF The incidence coalgebra of a (locally finite) simplicial set X is the k-vector space $C(X) = \bigoplus_{x \in X_1} k$

with $\Delta : C(X) \longrightarrow C(X) \otimes C(X)$

$$[x] \longmapsto \sum_{\sigma \in X} d_2\sigma \otimes d_0\sigma.$$

$\sigma \in X_2$
$d_1\sigma = x$



$\boxed{\text{Fact / "definition"}}$ $C(X)$ is a coassociative, counital k-coalgebra exactly when X is a **decomposition set** (aka a **2-Segal set**).

Gálvez-Carrillo – Kock – Tonks (and independently Dyckerhoff – Kapranov) generalize decomposition sets (resp.

generalize decompo... 2-Segal sets) to decomposition spaces (resp. 2-Segal spaces).

sets		spaces
$X : \Delta^{op} \to Set$	$\rightsquigarrow$	$X : \Delta^{op} \to S$ (spaces
vector space with basis $B \in Set$	$\rightsquigarrow$	objects in the slice cat. $S_{/B}$
$\mathcal{C}(X) = \bigoplus_{x \in X_1} k$	$\rightsquigarrow$	$\mathcal{C}(X) = S_{/X_1}$ "free vector space on X_1"
$\Delta : \mathcal{C}(X) \to \mathcal{C}(X) \otimes \mathcal{C}(X)$ $[x] \mapsto \sum_{d_1\sigma = x} d_2\sigma \otimes d_0\sigma$	$\rightsquigarrow$	$S_{/X_1} \to S_{/X_1 \times X_1}$ induced by

$$\begin{tikzcd}
 & X_2 \arrow[dl, "d_1"'] \arrow[dr, "(d_2, d_0)"] & \\
X_1 & & X_1 \times X_1
\end{tikzcd}$$

$$\zeta : [x] \longmapsto 1 \rightsquigarrow S_{/X_1} \xrightarrow{\zeta} S$$

induced by

$$\begin{tikzcd}
 & X_1 \arrow[dl, "id"'] \arrow[dr] & \\
X_1 & & *
\end{tikzcd}$$

$$\mu = \begin{cases} 0 & p^2 \mid n \\ (-1)^r & \end{cases}$$

$$= \Phi_{even} - \Phi_{odd}$$

$$\mu * \zeta = 1 \iff \boxed{\Phi_{even} * \zeta \simeq 1 + \Phi_{odd} * \zeta}$$

To recover the classical examples of zeta functions, take cardinality.

But there's so much more information available to us through decomposition spaces and homotopy theory!

Towards a decomposition space description of Kapranov's motivic zeta function

Recall: Kapranov's motivic zeta function for a k-variety X is the generating series

$$Z_{mot}(X,t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n \in K_0(\mathrm{Var}_k)[[t]]$$

$\mathrm{Sym}^n X = X^n / \Sigma_n$

Key fact: when X is a variety over $k = \mathbb{F}_q$,

$$Z(X,t) = \# Z_{mot}(X,t).$$

The Weil Conjectures imply $Z(-,t)$ extends to a motivic measure

$$\begin{aligned} K_0(\mathrm{Var}_{\mathbb{F}_q}) &\longrightarrow (1+t\mathbb{Z}_\ell[[t]], \cdot) \\ [X] &\longmapsto \prod_{i=0}^{2\dim X} \det\left(1-tF \mid H^i(X,\mathbb{Q}_\ell)\right)^{(-1)^{i+1}}. \end{aligned}$$

Campbell–Wolfson–Zakharevich construct a lift of this measure to their K-theory spectra:

$$\zeta : K(\mathrm{Var}_{\mathbb{F}_q}) \longrightarrow K(\mathrm{Aut}(\mathbb{Z}_\ell))$$

"derived ℓ-adic zeta function"

(taking π_0 recovers $Z(-,t)$)

Fact: $Z_{mot}(-,t)$ is also a motivic measure:

$$Z_{mot}(-,t) : K_0(\mathrm{Var}_k) \longrightarrow \left(1 + t\widetilde{K_0}(\mathrm{Var}_k)[[t]], \cdot\right).$$

$[X] = [U][X \setminus U]$

$[\mathrm{Sym}^n X] = \sum_{i+j=n} [\mathrm{Sym}^i U][\mathrm{Sym}^j V]$

invert finite radicial maps

$\boxed{\text{Big Idea \#1}}$ (Campbell-Zakharevich)

$Z_{mot}(-,t)$ should lift to a map

of K-theory spectra

$\int_{mot} : K(Var_k) \longrightarrow$???

$\boxed{\text{Big Idea \#2}}$ (very much in progress w/ Bergner, Feller)

$Z_{mot}(-,t)$ is the generating series of the zeta function in the incidence algebra of the decomposition space

Campbell '17 $\quad \tilde{S}_\bullet(Var_k) \to$ ~~SW cat.~~ CGW cat.

$A \rightsquigarrow S_\bullet(A) \rightsquigarrow K(A) = |\Omega S_\bullet A|$

$$U \xhookrightarrow{\text{op.}} X \xhookleftarrow{\text{cl.}} V$$

Some evidence for this approach:
(not totally baseless speculation, I promise!)

- It works for $X = \mathrm{Spec}\, k$:

  the incidence algebra zeta function
  is represented by the span

$$\mathrm{iso}(\mathrm{Var}_k) \xleftarrow{=} \mathrm{iso}(\mathrm{Var}_k) \longrightarrow *$$

which loosely corresponds to

$$[X] \longmapsto 1 \quad \text{for all } X$$

$$1 = [\mathrm{Spec}\, k]$$

On the other hand,

$$Z_{mot}(\operatorname{Spec} k, t) = (1-t)^{-1}$$

$$= \sum_{n=0}^{\infty} 1\, t^n$$

- For any $X \in Var_k$, there is a product formula

$$Z_{mot}(X,t) = \prod_{x} (1-t)^{-1} = \prod_{x} Z_{mot}(\operatorname{Spec} k(x), t)$$

so maybe we should really be treating $Z_{mot}(X,t)$ as a "relative zeta function" with respect to $X \to \operatorname{Spec} k$.

Hodge poly. of $X/\mathbb{C}$

Euler char.

$(k=\mathbb{C})$ Eule… of $X(\mathbb{C})$

$(k=\mathbb{C})$

$\tilde{S}_0(\mathrm{Var}_k)$

$(k=\mathbb{F}_q)$ derived ℓ-adic

motivic

arithmetic $(k=\mathbb{Q})$

$$\begin{array}{ccc}
K(\mathrm{Var}_{\mathbb{F}_q}) & & K(\mathrm{Var}_k) \\
\downarrow \zeta & \overset{\#}{\leftsquigarrow} & \downarrow \zeta_{mot} \\
K(\mathrm{Aut}(\mathbb{Z}_\ell)) & & K(K(\mathrm{Var}_k))
\end{array}
\qquad
\begin{array}{c}
K(\mathrm{Var}_{\mathbb{Q}}) \\
\downarrow \zeta_{arith} \\
K(\mathrm{Aut}(\mathbb{Z}))
\end{array}$$

$\boxed{\text{Questions?}}$

Bonus: there is a motivic Möbius function

$$M_{mot}(X,t) = Z_{mot}(X,t) = \prod_{x\in X}(1-t)$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n [\mathrm{Conf}^n X] t^n$$